Phase transitions in the one-dimensional frustrated quantum XY model and Josephson-junction ladders

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A one-dimensional quantum version of the frustrated XY (planar rotor) model is considered which can be physically realized as a ladder of Josephson junctions at half a flux quantum per plaquette. This system undergoes a superconductor to insulator transition at zero temperature as a function of charging energy. The critical behavior is studied using a Monte Carlo transfer matrix applied to the path-integral representation of the model and a finite-size-scaling analysis of data on small system sizes. Depending on the ratio between the interchain and intrachain couplings the system can have single or double transitions which is consistent with the prediction that its critical behavior should be described by the two-dimensional classical XY-Ising model.

The two-dimensional frustrated classical XY model has attracted considerable attention recently.\(^1\)–\(^8\) It can be related to Josephson-junction arrays in a magnetic field, where it is expected to describe the finite-temperature superconductor to normal transition in arrays with half a flux quantum per plaquette.\(^9\) At low temperatures where capacitive effects dominate, the array undergoes a superconductor to insulator transition as a function of charging energy.\(^10\)–\(^13\) These charging effects arise from the small capacitance of the grains making up the array and leads to strong quantum fluctuations of the phase of the superconducting order parameter. The critical behavior is now described by a two-dimensional frustrated quantum XY model with a Hamiltonian\(^12\)

\[
H = \frac{E_c}{2} \sum_r \left( \frac{d}{d\theta_r} \right)^2 \sum_{\langle rr' \rangle} E_{rr'} \cos(\theta_r - \theta_{r'}) .
\]  

(1)

The first term in Eq. (1) describes quantum fluctuations induced by the charging energy \(E_c = e^2/C\) of a non-neutral superconducting grain located at site \(r\), where \(e\) is the electronic charge and \(C\) is the effective capacitance of the grain. The second term is the usual Josephson-junction coupling between nearest-neighbor grains. \(\theta_r\) represents the phase of the superconducting order parameter and the couplings \(E_{rr'}\) satisfy the Villain’s “odd rule” in which the number of negative bonds in an elementary cell is odd.\(^14\) In a square lattice this can be satisfied, for example, by ferromagnetic horizontal rows and alternating ferromagnetic and antiferromagnetic columns of bonds. This rule is a direct consequence of the constraint that, for the half-flux case, the line integral of the vector potential due to the applied magnetic field should be equal to \(\pi\) in units of the flux quantum.

In this work we consider a one-dimensional frustrated quantum XY (1D FQXY) model\(^15\) which can be regarded as the simplest 1D version of the model (1) consisting just of a single column of frustrated plaquettes as indicated in Fig. 1. This can be physically realized as a periodic Josephson-junction ladder at half a flux quantum per plaquette.\(^16\)\(^,\)\(^17\) In the classical limit \((E_c = 0)\), the ground state of the 1D FQXY model has a discrete \(Z_2\) symmetry associated with an antiferromagnetic pattern of plaquette chiralities \(x_p = \pm 1\), as indicated in Fig. 1, measuring the two opposite directions of the supercurrent circulating in each plaquette. For small \(E_c\), there is a gap for creation of kinks in the antiferromagnetic pattern of \(x_p\), and the ground state has long-range chiral order.

Within a path-integral approach,\(^16\) it can be shown that the effective action describing quantum fluctuations in the

![FIG. 1. Schematic representation of the one-dimensional frustrated quantum XY model with inter- \((E_x)\) and intrachain \((\pm E_x)\) coupling constants. The antiferromagnetic ordering of chiralities \(x_p = \pm 1\) is also indicated.](http://jap.aip.org/jap/copyright.jsp)
1D FQXY model leads to two coupled XY models in two (space-time) dimensions which is expected to have a critical behavior in the universality class of the 2D XY-Ising model defined by the classical Hamiltonian

$$\beta H = -\sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \left[ A(1+\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\cos(\mathbf{\theta}_{\mathbf{r}}-\mathbf{\theta}_{\mathbf{r}'}) + C\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'} \right].$$

(2)

The phase diagram of this model consists of three branches, in the ferromagnetic region. One of them corresponds to single transitions with simultaneous loss of XY and Ising order. Further away from the branch point, this line of single transitions becomes first order. The other two lines correspond to separate XY and Ising transitions. The 1D FQXY model is represented by a particular path through this phase diagram which will depend on the ratio $E_x/E_y$ between the interchain and intrachain couplings of the model. In this work we describe the results of a finite-size scaling analysis of extensive calculations on the 1D FQXY model which shows that, in fact, the XY and Ising-like excitations of the quantum model decouple for large interchain couplings, giving rise to pure Ising model critical behavior for the chirality order parameter. As a consequence, the universality class of the superconductor-insulator transition in the related Josephson-junction ladder should then depend on the ratio between interchain and intrachain Josephson couplings.

To study the critical behavior of the 1D FQXY model, we find it convenient to use an imaginary-time path-integral formulation of the model. In this formulation, the one-dimensional quantum problem maps into a 2D classical statistical mechanics problem where the ground state energy of the quantum model of finite size $L$ corresponds to the reduced free energy per unit length of the classical model defined on an infinite strip of width $L$ along the imaginary time direction, where the time axis $\tau$ is discretized in slices $\Delta \tau$. After scaling the time slices appropriately in order to get a space-time isotropic model one obtains a classical partition function where the parameter $\alpha=(E_x/E_y)^{1/2}$ plays the role of an inverse temperature in the 2D classical model. The scaling behavior of the energy gap for kink excitations (chiral domain walls) of the 1D FQXY corresponds to the free energy of an infinite strip in this classical model. For large $\alpha$ (small charging energy $E_x$), there is a gap for creation of kinks in the antiferromagnetic pattern of $X_{\mathbf{r}}$, and the ground state has long-range chiral order. At some critical value of $\alpha$, the chiral order is destroyed by kink excitations, with an energy gap vanishing as $|\alpha-\alpha_c|^\nu$, which defines the correlation length exponent $\nu$. Right at this critical point, the correlation function decays as a power law $\langle X_{\mathbf{r}}X_{\mathbf{r}'} \rangle = |\rho - \rho'|^{-\eta}$ with a critical exponent $\eta$. The free energy per unit length $f(\alpha)$ of the Hamiltonian on the infinite strip can be obtained from the largest eigenvalue $\lambda_0$ of the transfer matrix between different time slices as $f = -\ln \lambda_0$. To obtain $\lambda_0$, we used a Monte Carlo transfer-matrix method which has been shown to lead to accurate estimates of the largest eigenvalue even for models with continuous symmetry. The implementation of this method is similar to the case of the 2D frustrated classical $XY$ model and further details can be found in that work.

The interfacial free energy for domain walls can be obtained from the differences between the free energies for the infinite strip with and without a wall. To obtain the critical exponents and critical coupling we employ the finite-size scaling

$$\Delta F(\alpha,L) = A L^{1/\nu} \delta \alpha,$$

(3)

where $A$ is a scaling function. The interfacial free energy for domain walls can be obtained from the differences between the free energies for the infinite strip with and without a wall. To obtain the critical exponents and critical coupling we employ the finite-size scaling

$$S = \partial \Delta F(\alpha,L)/\partial \alpha$$

evaluated near the critical coupling $\alpha_c$. The slope of the straight line gives an estimate of $1/\nu$.

Table II. Critical couplings $\alpha_c = (E_x/E_y)^{1/2}$ and critical exponents $(\nu, \eta)$, obtained from finite-size scaling analysis of interfacial free energies for two different values of the ratio $E_x/E_y = 1$ and 3 are indicated in Table I.

<table>
<thead>
<tr>
<th>$E_x/E_y$</th>
<th>$\alpha_c$</th>
<th>$\nu$</th>
<th>$\eta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.04(1)</td>
<td>0.81(4)</td>
<td>0.47(4)</td>
</tr>
<tr>
<td>3</td>
<td>1.16(2)</td>
<td>1.05(6)</td>
<td>0.27(3)</td>
</tr>
</tbody>
</table>
exponents ($\nu=1, \eta=0.25$). This result points to a single transition scenario. In fact, they are consistent with a point along the line of single transitions in the $XY$-Ising model.\(^\text{18}\) It is interesting to note that this result is also consistent with similar calculations for the 2D frustrated classical $XY$ model.\(^\text{8}\) In general, the critical behavior of a $d$ dimensional quantum model is in the same universality class of the $d+1$ dimensional classical version. However, the 1D $FQXY$ model, apparently, is not the Hamiltonian limit of the 2D classical model. Yet, their critical behavior appears to be in the same universality class.

For the case of unequal couplings $E_x/E_y=3$, the results indicated in Table I are in good agreement with pure 2D Ising values. From the relation between the 1D $FQXY$ model and the 2D $XY$-Ising model this then implies that the $XY$ and Ising like excitations have decoupled in this region of parameters. To show that in fact one has two decoupled and, at the same time, separated transitions we show in Fig. 3 the results for the helicity modulus which measures the response of the system to an imposed twist. The helicity modulus is related to the free-energy differences $\Delta F$ between strips with and without any additional phase mismatch of $\pi$ along the strip and is given by $\gamma=2\Delta F/\pi^2$ for large system sizes. If the model is decoupled then the transition should be in the universality class of the 2D classical $XY$ model, where one knows that the transition is associated with a universal jump of $2/\pi$ in the helicity modulus.\(^\text{23}\) The critical coupling can be estimated as the value of $\alpha$ at which $\Delta F=\pi$ which gives $\alpha_c=1.29$. This is to be compared with the critical coupling for the destruction of chiral order in Table I, $\alpha_c=1.16$. This clearly indicates the transitions are well separated and thus one expects they are decoupled. We have also performed less detailed calculations at other values of the ratio $E_x/E_y$ from which we can estimate that the Ising and $XY$ transitions merge into a single transition roughly at $E_x/E_y\approx 2$. Since, the superconductor to insulator transition is to be identified with the loss of phase coherence\(^\text{24}\) we reach the interesting result that in the 1D $FQXY$, or alternatively, a Josephson-junction ladder, the universality class of the superconductor-insulator transition depends on the ratio between intra- and interchain couplings.

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