THIRD-BODY PERTURBATION USING A SINGLE AVERAGED MODEL: APLICATION TO RETROGRADE ORBITS

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Abstract. *In the literature, there are several papers studying the effects of the third body perturbation in spacecrafts. Most of them work with the Hamiltonian of the system or with the disturbing function expressed in an analytic manner. The present paper has the goal of developing a semi-analytical study of the perturbation caused in a spacecraft by a third body using a single averaged model to eliminate the terms due to the short time periodic motion of the spacecraft. To perform this task, the potential is expanded in Legendre polynomials up to fourth order. After that, the averaged potential is used to obtain the equations of motion of the spacecraft, using the planetary equations due to Lagrange. Then, many trajectories are computed by numerical integration of the equations of motion and several plots show the time histories of the Keplerian elements of the orbits involved. One of the most important applications is to calculate the effect of Lunar and Solar perturbations on high-altitude Earth satellites. In particular, our semi-analytical model is applied to study the effect of the Moon in spacecrafts that are in retrograde orbits around the Earth.*

***Keywords**: astrodynamics, third-body perturbation, celestial mechanics, spacecrafts.*

1. Introduction

The effects of the gravitational attractions due to the Sun and the Moon in the orbits of Earth's artificial satellites have been studied in many papers. Kozai (1959) writes down Lagrange's planetary equations and the disturbing function due to the Sun or the Moon, including both secular and long periodic terms, but it only gives explicit expressions for the secular terms. Blitzer (1959) ignores the specialized techniques of celestial mechanics and obtains estimates for the perturbations by using methods of classical mechanics. Again, only secular terms are included, and it shows that the principal effect of the third body perturbation is a precession of the orbital plane around the pole of the ecliptic. Musen (1961) shows two systems of formulas for the determination of the long periodic perturbations. The first system uses the theory originally developed by Gauss for a numerical treatment of the very long periodic effects in planetary motion, and the second method is based on the development of the disturbing function in terms of the Legendre polynomials and it finds the presence of long periodic terms and the influence on the stability of the orbit.

Cook (1962) studied the perturbations due solely to a third body from Lagrange's planetary equations by integrating over one revolution of the satellite. The rates of change of the orbital elements averaged over one revolution are then written and all the first order terms (secular and long-period) are retained in the analysis. The theory is limited to satellites whose semi-major axis does not exceed one tenth of the Earth-Moon distance.

After that, Giacaglia (1973) obtained the disturbing function for the perturbation of the Moon using ecliptic elements for the Moon and equatorial elements for the satellite. Secular, long-period and short-period perturbations are then computed, with the expressions kept in closed form in both inclination and eccentricity of the satellite. Alternative expressions for short-period perturbations of high satellites are also given, assuming small values of the eccentricity.

Hough (1981) used the Hamiltonian formed by a combination of the declination and the right ascension of the satellite, the Moon, and the Sun. After that he averaged the Hamiltonian in small and moderate fluctuations and studied periodic perigee motion for orbits near the critical inclinations 63.4^0 and 116.6^0. The theory predicts the existence of larger maximum fluctuations in eccentricity and faster oscillations near stable equilibrium points. Delhaise and Morbidelli (1993) investigated the Lunisolar effects of a geosynchronous artificial satellite orbiting near the critical inclination, analyzing each harmonic formed by a combination of the satellite longitude of the node and the Moon's longitude of the node. He demonstrates that the dynamics induced by these harmonics does not show resonance phenomena.

Other researches developed by Broucke (1992), Prado and Costa (1998) and Prado (2003) show general forms of the disturbing function of the third body truncated after the term of second and fourth order, in the expansion in Legendre polynomials. After that, Costa (1998) expanded the order of this model to order eight.

In the present paper several topics related to the third body perturbation problem are studied. In particular, the so called "critical angle of the third-body perturbation," which is a value for the inclination that separates stable and unstable orbits. The motion of the spacecraft is studied under the single-averaged analytical model with the disturbing function expanded in Legendre polynomials up to fourth-order. The single-average is taken over the mean motion of the satellite. After that, the equations of motion are obtained from the planetary equations (Taff, 1985) in an analytical format. Those equations are then integrated numerically using the Runge-Kutta of order 7-8 in a PC computer in FORTRAN language. The fact that modern computers can easily integrate numerical trajectories using complex models for the dynamics does not invalidate the use of models based in analytical approximations. The most important reason is that a single-averaged model, like the one shown here, can eliminate short-period periodic perturbations that appear in the trajectories. In that way, smooth curves that show the evolution of the mean orbital elements for longer time periods can be constructed, which give a better understanding of the physical phenomenon studied and allow the study of long-term stability of the orbits in the presence of disturbances that cause slow changes in the orbital elements. Note also that the truncated equations of motion can be numerically integrated much faster than the full equations of the restricted three-body problem.

2. Mathematical Model

The dynamical model used in the present paper can be formulated in a very similar way of the formulation of the planar restricted three-body problem:

- There are three bodies involved in the dynamics: one body with mass $\mathbf{m_0}$ fixed in the origin of the reference system, a second massless body in a three-dimensional orbit around $\mathbf{m_0}$ and a third body in a circular orbit around $\mathbf{m_0}$ (see Figure 1).
- The motion of the spacecraft (the second massless body) is Keplerian and three-dimensional, with its orbital elements perturbed by the third body involved in the dynamics. The motion of the spacecraft is studied with the single averaged model, where the average is performed with respect to the true anomaly of the spacecraft (**f**). The disturbing function is then expanded in Legendre polynomials to generate the disturbing potential.

This section derives the equations required to perform the simulations. The main body $\mathbf{m_0}$ is fixed in the center of the reference system X-Y. The perturbing body **m'** is in a circular orbit with semi-major axis **a'** and mean motion **n'** (from the Law of Kepler: $n'^2 a'^3 = G(m_0 + m')$). The spacecraft is in a three dimensional orbit, with orbital elements **a**, **e**, **i**, ω, Ω, and mean motion **n** (from the Law of Kepler: where $n^2 a^3 = Gm_0$).

In this situation, the disturbing potential that the spacecraft has from the action of the perturbing body is given by (using the expansion in Legendre polynomials and assuming that r' >> r) (Broucke, 1992):

$$R = \frac{\mu' G(m_0 + m')}{\sqrt{r^2 + r'^2 - 2rr' Cos(S)}} = \frac{\mu'(m_0 + m')}{r'} \sum_{n=2}^{\infty} (\frac{r}{r'})^n P_n Cos(S) \tag{1}$$

The parts of the disturbing potential due to P_2 to P_4 (that are the parts considered in the present paper) are:

$$R_2 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^2 P_2 Cos(S) \quad (2)$$

$$R_3 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^3 P_3 Cos(S) \quad (3)$$

$$R_4 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^4 P_4 Cos(S) \quad (4)$$

The next step is to average those quantities over the short period of the satellite. The definition for average used in the present paper is:

$$< G >= \frac{1}{2\pi}\int_0^{2\pi} G dM \quad (5)$$

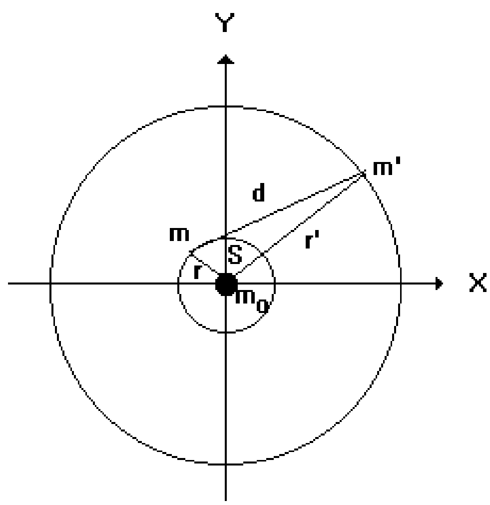

Fig. 1. Illustration of the third body perturbation.

Figure 2 shows the relations between the orthogonal set of vectors $\hat{P}$, $\hat{Q}$, $\hat{R}$ and the coordinate system that lies on the orbital plane. $\hat{r}'$ and $\hat{r}$ are unit vectors pointing from the central body to the perturbing body and from the central body to the satellite, respectively. M is the mean anomaly of the satellite and M' is the mean anomaly of the perturbing body. The results are valid for the special case when the perturbing body is in a circular orbit and the initial mean anomaly of the perturbing body is equal to zero. The following relations are available (Broucke, 1992):

$$\alpha = Cos(\omega)Cos(\Omega - M') - Cos(i)Sin(\omega)Sin(\Omega - M') \quad (6)$$

$$\beta = -Sin(\omega)Cos(\Omega - M') - Cos(i)Cos(\omega)Sin(\Omega - M') \quad (7)$$

With those relations it is possible to relate the angle S with the positions of the perturbing and the perturbed bodies. It is done using the equation:

$$Cos(S) = \alpha Cos(f) + \beta Sin(f) \quad (8)$$

Then, it is possible to perform the process of average. The results available are:

$$< R_2 >= \frac{\mu' a^2 n'^2}{2}\left(\frac{a'}{r'}\right)^3\left(\left(1+\frac{3}{2}e^2\right)\left(\frac{3}{2}(\alpha^2+\beta^2)-1\right)+\frac{15}{4}e^2(\alpha^2-\beta^2)\right) \tag{9}$$

$$< R_3 >= \frac{\mu'(a)^3(n')^2}{2a'}\left(\frac{a'}{r'}\right)^4\left[\frac{15\alpha e(4+3e^2)}{8}-\frac{25\alpha^3 e(3+4e^2)}{8}+\frac{75\beta^2(e^2-1)\alpha e}{8}\right] \tag{10}$$

$$\begin{aligned} < R_4 >= {} & \frac{3\mu'(a')^3(a)^4(n')^2}{64(r')^5}((8+40(e)^2)+15(e)^4)-10\alpha^2(4+41(e)^2+18(e)^4) \\ & +35\alpha^4(1+12(e)^2+8(e)^4)-10\beta^2(4-(e)^2-3(e)^4)+70\alpha^2\beta^2(1+5(e)^2-6(e)^4) \\ & +35\beta^4((e)^2-1)^2) \end{aligned} \tag{11}$$

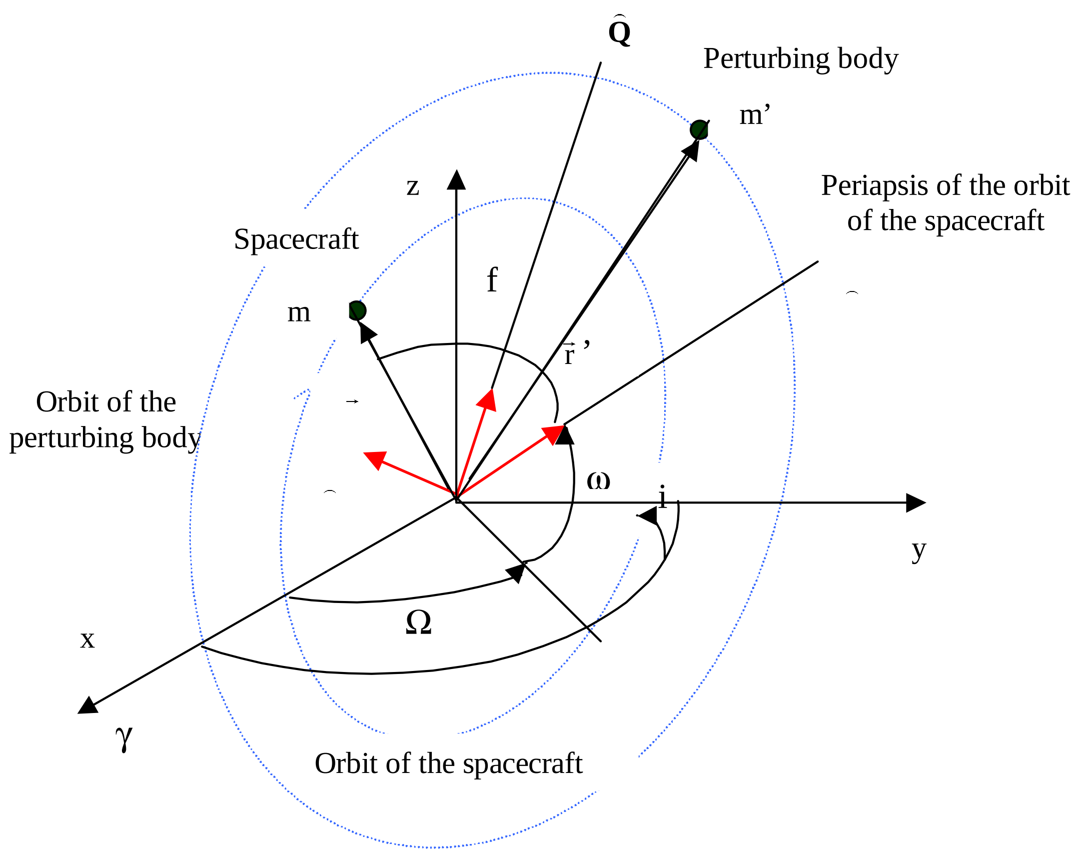

Fig. 2. The orthogonal set $\hat{\mathbf{P}}$, $\hat{\mathbf{Q}}$, $\hat{\mathbf{R}}$ and the coordinate system in the orbital plane.

The next step is to obtain the equations of motion of the spacecraft. This task is performed by using the Lagrange's planetary equations, that is a set of equations that provides the equations of motion based on the derivatives of the disturbing function (see Taff, 1985). It is noticed that the semi-major axis always remains constant. This occurs because, after the averaging, the disturbing potential does not depend on M_0 and, in consequence, the derivative of the potential with respect to this variable is zero.

RESULTS

The initial orbital elements used for the simulations are: a[0] = 0.341, e[0] = 0.01, Ω[0] = 0, ω[0] = 0, M[0] = 0. The time is defined such that the period of the disturbing body is 2π (canonical system of units). The plots of the time evolution of the orbital elements are shown for several values of the initial inclination. Comparing those results with the ones available in the literature (Prado, 1993; Prado and Costa, 1998); Costa (1998); Prado (2003) it is visible that the evolution of the inclination for retrograde orbits show a behavior that looks like a mirror image of the results available for the direct orbits. Figure 3 shows the evolution of the inclination. It is possible to identify a periodic behavior for values of the initial inclination equals to 100 deg, 120 deg and 140 deg. For values above 141 deg, that is near to the supplement of the critical angle of the third body perturbation, the inclination show straight lines.

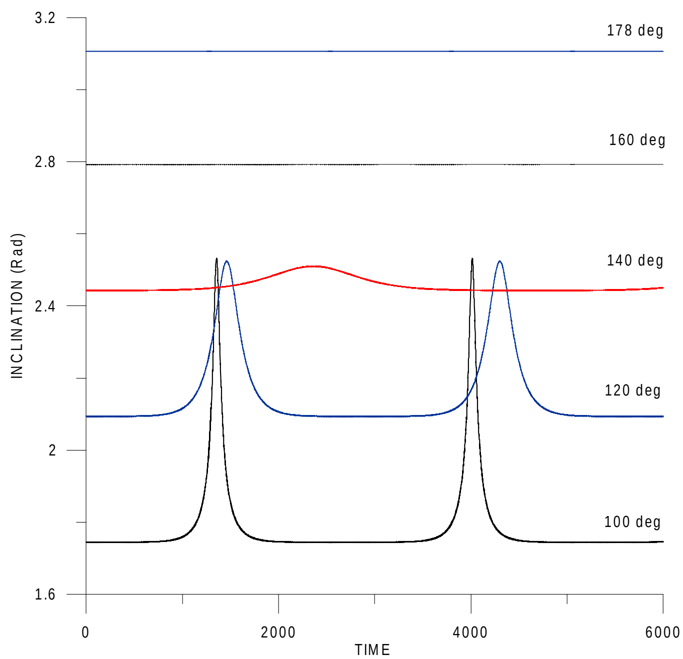

Figure. 3- Evolution of the inclination.

After that, we study the time evolution of the eccentricity. It is visible that the eccentricity oscillates with very small amplitude for values of the initial inclination above the supplement of the critical angle of the third body perturbation. For values of the initial inclination below this limit, there are larger amplitudes (see Figure 4 and 5). The argument of pericenter shows a secular and oscillating behavior for values of the initial inclination above 90 deg (see Figure 6).

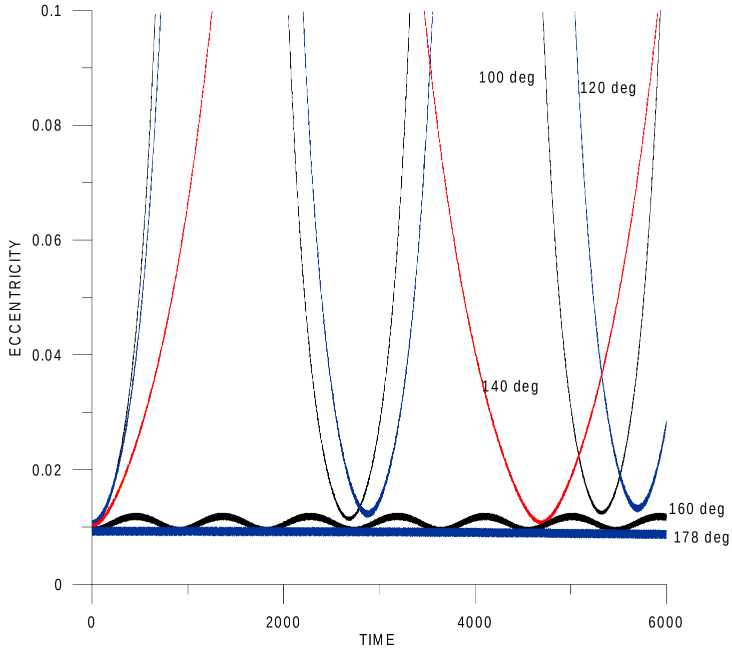

Figure. 4- Evolution of the eccentricity.

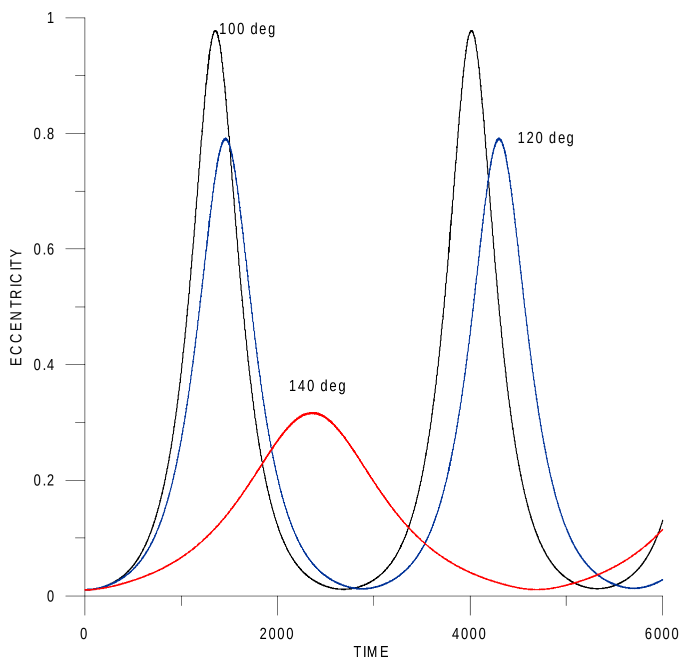

Figure. 5- Evolution of the eccentricity.

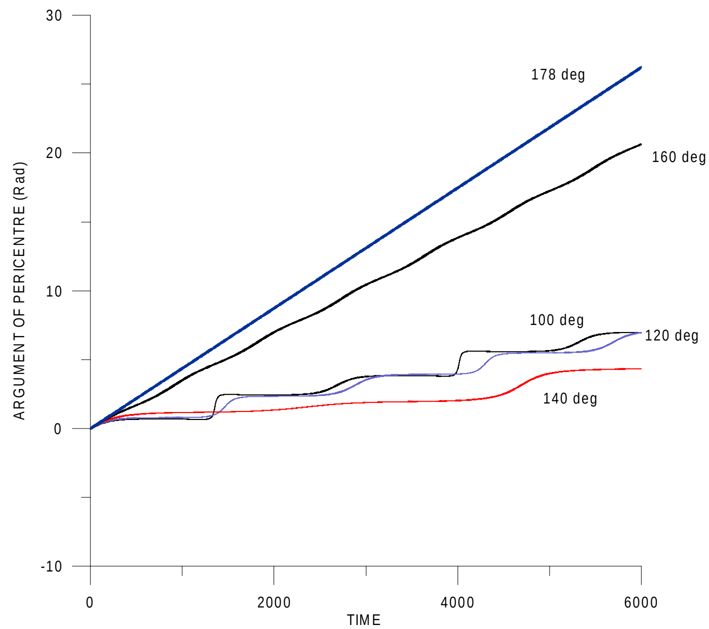

Figure. 6- Evolution of the argument of pericenter.

Looking in Fig. 6, it is visible that the oscillatory behavior of the argument of the pericenter disappears for values of the initial inclination near the supplement of the critical angle. Remember that, for short times, the amplitude is very small. The evolution of the longitude of the ascending node (see Figure 7) for retrograde orbits shows that it undergoes to positive increase (for direct orbits there is a regression).

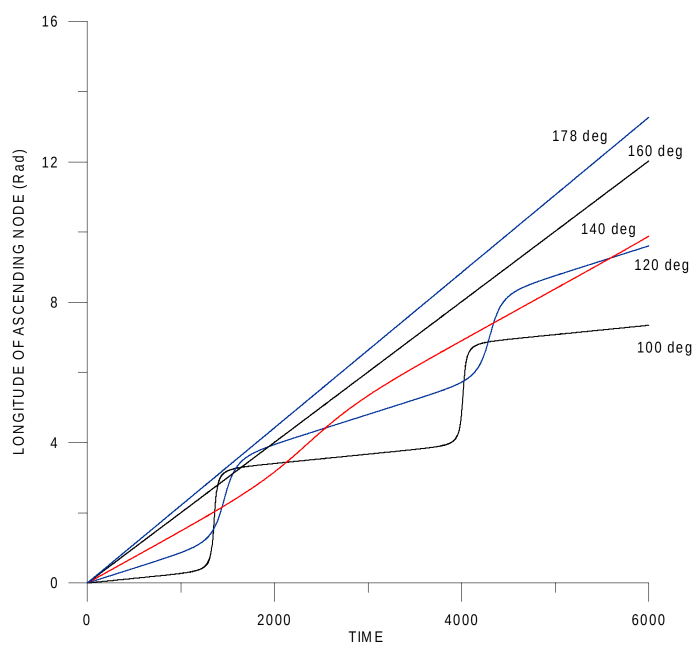

Figure. 7- Evolution of the longitude of ascending node.

The results for situations where the initial inclination is below the supplement of the critical angle have a characteristic behavior. The inclination starts with an initial value and, when it reaches the supplement of the critical angle, it reverses the rate of change and starts to decrease until the initial inclination value is reached again. Then, the period starts again and the inclination keeps oscillating between the initial value and the supplement of the critical angle. Looking at the evolution of the inclination vs. eccentricity, it is noticed that, when the inclination reaches it maximum value, the eccentricity reaches also its maximum value. It is also visible that this curve is periodic (see Figure 8). These curves are inverted, when compared to the ones that come from direct orbits and they show a behavior similar to a specular image.

It is possible to understand those behaviors by examining the second-order equations of motion, which adequately represents the system. The magnitude of the time derivative of the eccentricity is dependent on the term $e\sqrt{1-e^2}$, so it increases when the eccentricity has lower values due to the presence of e but, after this maximum value is reached, it starts to decrease due to the term $\sqrt{1-e^2}$. Its sign is determined exclusively by sin(ω) and it imposes the oscillatory behavior shown in the plots, since ω has a secular variation. For the time derivative of the inclination the same analysis can be applied. The magnitude is influenced by the term $e^2/\sqrt{1-e^2}$, which causes the curious behavior of the existence of regions of almost constant inclination alternated with sharp decreases and increases. The explanation is that, for lower values of the eccentricity, the term e^2 forces the time derivative to stay close to zero and the inclination remains almost constant. When the eccentricity increases and reaches values close to 1.0, the term $\sqrt{1-e^2}$ present in the denominator forces the time derivative to decrease. The alternation of the sign is caused exclusively by the term sin(ω), as explained before. The argument of the pericenter has a secular variation, because its time derivative is always positive in the situations considered, alternating large regions of slow increase with short regions of sharp oscillations. The sharp increase is also explained by the term $\sqrt{1-e^2}$ in the denominator that goes close to zero when the eccentricity approaches 1.0. The critical angle is important for stability of orbits near-circularity.

The practical application of those results is that only near-circular orbits with inclination higher than the critical value are stable in the long range, since for values of the initial inclination below this value the orbit looses its characteristics of near-circularity.

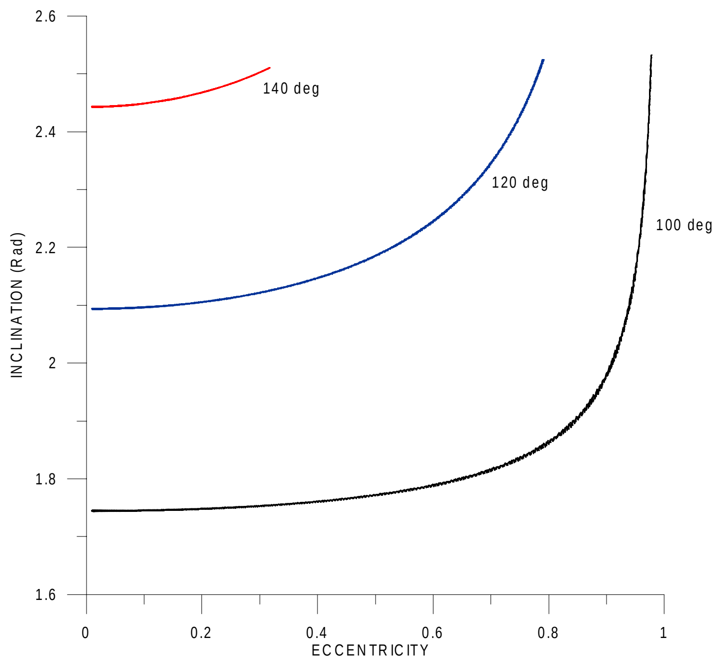

Figure. 8- Evolution of inclination vs eccentricity.

CONCLUSIONS

This paper developed analytical equations that can compute the effects of the third body perturbation in satellites around the Earth using a single averaged model. The disturbing potential is obtained and expanded up to the fourth order. After that, the equations of motion are obtained and applied in the important case of retrograde orbits. The results show the time evolution of the keplerian elements of the orbits considered, as a function of the initial inclination. The orbital elements present small oscillations and/or secular behavior. This semi-analytical model is able to study the evolutions of the orbital elements and the importance of the critical inclination in the stability of near circular orbits. The results for the retrograde orbits show that there is a mirror effect with respect to the results obtained for the case of direct orbits. Several plots showed the time histories of the Keplerian elements of the orbits studied under this model.

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