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ABSTRACT

We address the problem of synchronization in nonhyperbolic systems. We present a general method based on a chaos control strategy and on the Extended Kalman Filter that works for all classes of those systems and even in the presence of noise. The method works fine even for systems that present the phenomena of Unstable Dimension Variability (UDV).

INTRODUCTION

The inherent sensitive dependence on initial conditions is a fundamental characteristic of chaotic systems (Devaney, 1989). It implies that two trajectories starting from slightly different initial conditions diverge exponentially in time. Despite that, since 1983 it has been known that two chaotic systems can be synchronized (Fujisaka and Yamada, 1983; Afraimovich et al, 1986). Later on, Pecora and Carol (Pecora and Carroll, 1990) gave a condition for the synchronization of two identical chaotic systems: Using appropriately chosen state variables of a chaotic system (the driver) as input to a replica of the original system, the replica subsystem (the slave) might synchronize with the original system if its Lyapunov exponents are all negative. The authors also pointed out that this phenomenon of chaos synchronization may be used as a new paradigm to be implemented in communication systems. Since that work, synchronization in chaotic systems has become an area of intense activity (Parlitz et al, 1992; Cuomo and Oppenheim, 1992; Cuomo et al, 1993; Chua et al, 1993; Heagy et al, 1994; Kocarev and Parlitz, 1996; Pecora et al, 1997; Fink et al, 2000).

In this work, we discuss another approach for synchronization. This approach uses a scalar transmitted signal and is based on the use of the Extended Kalman Filter (Kalman, 1960; Brown and Hwang, 1997) in association with an OGY inspired method of control of chaos (Ott et al, 1990; Grebogi and Lai, 1997;

Macau and Grebogi, 2000). As so, in the slave system, the current state of the driver system is obtained form the scalar transmitted signal by using the Extended Kalman Filter. Then, a perturbation that depends on the current states of the both system is applied which ends up with the slave system following the driver (Macau et al, 2002). This approach is called *active synchronization* to emphasize its main difference in relation to the more traditional passive synchronization process in which synchronization happens as a consequence of a proper coupling scheme. Furthermore, it works properly even on situations in which all other approaches for synchronization fail, as are the cases for systems under the occurrence of the phenomenon of the *Unstable Dimension Variability* (*UDV*).

In the next section we review the concepts about hyperbolicity and unstable dimension variability. After that, we discuss the fundaments about synchronization and control of chaos. Subsequently, we review the ideas about control of chaos, introduce our active synchronization method and present the main results.

HYPERBOLICITY AND THE CONCEPT OF UNSTABLE DIMENSION VARIABILITY

In what follows, we consider just discrete in time maps. This assumptions is, however, not so restrictive because those maps can also be regarded as invertible Poincaré maps of continuous in time flows. Let $\mathbf{f}:\Re^m \to \Re^m$ be a diffeomorphism $\mathbf{x} \to \mathbf{f}(\mathbf{x})$ having an invariant set Ω, such that if $\mathbf{x} \in \Omega$, then any subsequent forward or backward iterates of $\mathbf{x}$ remain in Ω. The invariant set Ω is called hyperbolic if the tangent space at a point $\mathbf{x} \in \Omega$, denoted T_x, may be decomposed as the direct sum (Devaney, 1989)

$$T_x = E_x^u \oplus E_x^s, \tag{1}$$

where E_x^u and E_x^s are the unstable and stable subspaces, respectively, having the following properties:

(a) The decomposition (1) varies continuously with $\mathbf{x} \in \Omega$, and it is invariant under the action of the tangent map such that

$$\boldsymbol{Df}\left(E_{\mathrm{x}}^{\mathrm{u}}\right) = E_{\mathrm{f(x)}}^{\mathrm{u}}, \tag{2}$$

$$\boldsymbol{Df}\left(E_x^s\right) = E_{f(x)}^s; \tag{3}$$

(b) There exist constants $K > 0$ and $0 < r < 1$, such that

$$\left\|\boldsymbol{Df}^n(\boldsymbol{x})\boldsymbol{y}\right\| < Kr^n\|\boldsymbol{y}\| \text{ if } \boldsymbol{y} \in \boldsymbol{E}_x^s, \tag{4}$$

$$\left\|\boldsymbol{Df}^{-n}(\boldsymbol{x})\boldsymbol{y}\right\| < Kr^n\|\boldsymbol{y}\| \text{if } \boldsymbol{y} \in E_x^u, \tag{5}$$

meaning that vectors in a small neighborhood of $E_x^s\left(E_x^u\right)$ under forward (backward) iterations of the tangent map $\boldsymbol{Df}$ approaches any $\mathbf{x} \in \Omega$ at a uniform rate r. The *stable* (*unstable*) *dimension* at a point $\mathbf{x} \in \Omega$ is the dimension of the corresponding stable (unstable) subspace $d^s = \dim\left(E_x^s\right)$ ($d^u = \dim\left(E_x^u\right)$). For the invariant set Ω of the nonlinear map $\boldsymbol{f}(\boldsymbol{x})$, the stable $W^s(\boldsymbol{x})$ and the unstable $W^u(\boldsymbol{x})$ manifolds of the fixed point $\mathbf{x} \in \Omega$ are defined as (Devaney, 1989)

$$W^s(\boldsymbol{x}) = \left\{\boldsymbol{y} \in \Re^m : \boldsymbol{f}^n(\boldsymbol{y}) \to \boldsymbol{x} \quad \text{if} \quad n \to \infty\right\}, \tag{6}$$

$$W^u(\boldsymbol{x}) = \left\{\boldsymbol{y} \in \Re^m : \boldsymbol{f}^{-n}(\boldsymbol{y}) \to \boldsymbol{x} \quad \text{if} \quad n \to \infty\right\}, \tag{7}$$

respectively.

Due to the local manifold theorem (Devaney, 1989) for a hyperbolic C^r-diffeomorphism, the stable and unstable manifolds are tangent to the stable and unstable invariant subspace of the tangent map $\boldsymbol{Df}(\boldsymbol{x})$ at the fixed points embedded in Ω. Moreover, in hyperbolic systems the unstable and stable manifolds intersect transversely, i. e., the angle between them is bounded away from zero. Otherwise, the invariant set Ω is *nonhyperbolic*.

Nonhyperbolic systems encompass the majority of dynamical systems of physical and technological interest, and can be classified in two types. For the first type, the splitting of the phase space into expanding and contracting subspaces is invariant along a trajectory except at the tangencies of the stable and unstable manifolds., where the angles between subspaces are zero (Hammel et al, 1987; Grebogi et al, 1990; Sauer and Yorke, 1991).

The second type of nonhyperbolicity is the due to the presence of the a phenomena known as *unstable dimension variability* (*UDV*) (Abraham and Smale, 1970; Dawson et al, 1994; Dawson, 1996; Kostelich et al, 1997; Lai et al, 1999). It is related to the presence of unstable periodic orbits with different numbers of unstable directions embedded within the chaotic attractor. The sets of periodic orbits are densely mixed so that a typical trajectory experiences different numbers of unstable and stable directions as it evolves. As a consequence, the continuous splitting of the phase space into expanding and contracting subspaces is no longer valid. The unstable dimension variability is reflected by a finite-time Lyapunov exponent fluctuating about zero due to visits of the trajectory to regions of the attractor with a varying number of stable and unstable directions (Dawson et al, 1994; Sauer et al, 1997; Pikovsky et al, 1997).

UDV has been first described by Abraham and Smale (Abraham and Smale, 1970) for a diffeomorphism in $T^2 \times S^2$ whose invariant set has two fixed points with different unstable dimensions. However, the first observation of UDV for a dynamical system of physical interest appears in 1992, when its existence was incidentally reported for the kicked double rotor, a four-dimensional invertible map (Romeiras et al, 1992). This system consists of two connected massless rods. The first rod rotates about a fixed pivot; the second rod pivots about the ends of the second rod at a constant interval. This system is modeled by the following four-dimensional map, which describes the dynamics of the rotor relating the state of the system just after consecutive kicks:

$$\begin{aligned} \boldsymbol{T}_{i+1} &= \boldsymbol{MF}_i + \boldsymbol{T}_i, \\ \boldsymbol{F}_{i+1} &= \boldsymbol{LF}_i + \boldsymbol{G}(\boldsymbol{T}_{i+1}), \end{aligned} \tag{8}$$

where

$$\boldsymbol{T} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in S^1 \times S^1, \quad \boldsymbol{F} = \begin{pmatrix} x(3) \\ x(4) \end{pmatrix} \in \Re \times \Re, \tag{9}$$

end

$$\boldsymbol{G}(\boldsymbol{T}) = \begin{pmatrix} c_1 sin(x(1)) \\ c_2 sin(x(2)) \end{pmatrix} \tag{10}$$

In Eqs. (8) to (10), $x(1,2)$ are the angular positions of the rotors ate the instant of the *i*th kick, while $x(3,4)$ are the angular positions of the rotors, $\boldsymbol{L}$ and $\boldsymbol{M}$ are 2×2 constant matrices whose elements depend on the physical parameters of the rotors, c_1 and c_2 are two parameters that are proportional to the kicking strength *f*. In what follows, we choose there parameters so that $c_1 = c_2 = f$, and we use for $\boldsymbol{L}$ and $\boldsymbol{M}$ the same values that appears in (Romeiras et al, 1992). Thus, the only parameter that can vary is the kicking strength *f*m which is used as the externally adjustable control parameter.

For the double-rotor mar, numerical experiments (Romeiras et al, 1992) show that the system goes through a cascade of period-doubling bifurcation for $f < f_1 \approx 6.75$ and becomes chaotic with one positive Lyapunov exponent at f_1. For values of *f* near $f = 8.0$, there is a transition from one positive Lyapunov exponent to two positive ones, while the finite-time exponents shows fluctuations (Dawnson et al, 1994; Lai et al, 1999) between one and two positive exponents. For parameter *f* values much above the the transition value, such as the case for $f = 9.0$, the second Lyapunov exponent becomes positive. Thus, there is unstable dimension variability for values of *f* near $f = 8.0$, and for $f = 9.0$ the double-rotor is a hyperchaotic system.

SYNCHRONIZATION IN CHAOTIC AND IN CHAOTIC AND NONHYPERBOLIC SYSTEMS

Under well know conditions, two chaotic systems can synchronize (Pecora and Carroll, 1990; Pecora and Carroll, 1991). Let us consider a transmitter system (driver) such as

$$\frac{dx}{dt} = f(x,y,z), \quad \frac{dy}{dt} = g(x,y,z), \quad \frac{dz}{dt} = h(x,y,z). \tag{11}$$

The receiver system has the same dynamics as the transmitter, and it receives as a signal one of the state space variables of the transmitter, say *y*, that act on the receiver as the driver. Note that there are many ways to apply this signal to the receiver. Thus, the receiver can be described by

$$\frac{dx'}{dt} = f(x',y,z'), \quad \frac{dz'}{dt} = h(x',y,z'), \tag{12}$$

where the primed variable are the response only and we have applied the drive only in the *y* of the response subsystem. It has been shown (Pecora and Carroll, 1990) that if the Lyapunov exponents in the receiver system are negative, then $y' - y \to 0$ as $t \to \infty$. This means that the chaotic evolution of the transmitter can be recovered by the receiver if just a proper chosen signal is transmitted over the communication channel.

In regard to synchronization in chaotic systems with more than one positive Lyapunov exponent (hyperchaotic systems), there was a general belief that to achieve synchronization the number of variables to be transmitted should be equal to that of positive Lyapunov exponents in order to account for the same number of unstable directions along the trajectory (Pyragas, 1993). Peng and coworkers (Peng et al, 1996) showed that this belief is incorrect and the synchronization could be possible, in some cases, using a signal constructed in the form of the linear combination of the original phase space variables. Subsequent works introduced modifications to Peng et al assertion, modifications related to the form of the transmitted signal should be constructed (Tamasevicius and Cenys, 1997), or to the parameters of the transmitted signal (Ali and Fang, 1997; Johnson et al, 1998), or to the coupling between the transmitted signal and the receiver (Pecora et al, 1997). In general, all those proposed approaches uses feedback strategies whose parameters are fixed and calculated using empirical strategies or optimization algorithms. As a consequence, none of those strategies can be considered to work for sure with any hyperchaotic system..

In all the ideas previously discussed, the requirement of a hyperbolic structure for the systems to be synchronized is implicit. If it is not the case, those approaches may not work. Thus, the presence of tangencies can cause the synchronized systems to lose the synchronization near a tangency point. It happens because, under the presence of noise or parameter mismatch, one of the trajectories can be pushed across the stable manifold, while the other stays on the unstable set. If this happens, the trajectories tend to separate exponentially on average from each other and so the synchronization is destroyed. The situation is even worst for systems having unstable dimension variability and in the presence of noise or even small parameter mismatches . As the trajectories move from one neighborhood to another having unstable periodic orbits with different number of expanding directions the synchronized trajectories tend to separate exponentially from each other and so the synchronization is destroyed. What makes the situation hard for this case of nonhyperbolicity is the fact that the sets of periodic orbits with different number of expanding directions are densely mixed (Kostelich et al, 1997). As a consequence, the regions where the synchronization is highly susceptible to being destroyed due to the presence of noise are extended over most of the attractor. In this scenario, the proposed methods for synchronization of hyperchaotic systems are extremely inefficient. As a consequence, another efficient approach is required.

EXTENDED KALMAN FILTER AND ACTIVE SYNCHRONIZATION

For nonhyperbolic hyperchaotic systems, a synchronization strategy will work only if it is capable of continuously keeping track of the local changes of the system as the trajectory evolves through the sets of periodic orbits with different number of expanding directions and changes its parameters accordingly. Thus, the main point here is to have a synchronization procedure with a built-in adjustment mechanism which monitors the system local dynamics and adjust its coefficients to keep the systems synchronized. Herewith we propose a synchronization method based on a chaos control strategy that implements exactly this principle.

A schematic illustration of our method to actively synchronize tow chaotic systems is showed in Fig. 1. We extend the pole placement control of chaos strategy (Romeiras et al, 1992) to stabilize a chaotic strategy of one system (*System B*) about a chaotic orbit of the other system (*System A*) to achieve synchronization of the two systems. We assume that some parameter of the system can be externally adjustable, and that we have complete access to the state variable of the slave system (*System B*). The Extended Kalman Filter allows us to estimate the current state of the driver system with the

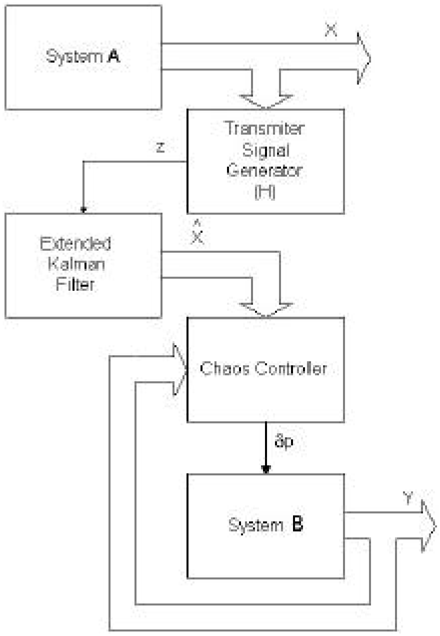

Fig1: Schematic illustration of our strategy of active synchronization. Based on measures from the state variables of System A the parameter p of System B is adjusted so that it is kept synchronized with System A. The current state of the driver System A is obtained from the scalar transmitted signal z by using the Extended Kalman Filter. dp is the small perturbation applied on some parameter of the System B to make it synchronize with System A.

transmitted scalar signal. Let us consider two almost identical chaotic systems that are described by the following maps on the Poincaré surface of section:

$$\textit{System A: } \boldsymbol{Y}_{i+1} = \boldsymbol{F}(\boldsymbol{Y}_i, \overline{p}), \tag{13.a}$$

$$\textit{System B: } \boldsymbol{X}_{i+1} = \boldsymbol{F}(\boldsymbol{X}_i, p), \tag{13.b}$$

where $\boldsymbol{X}_i$, $\boldsymbol{Y}_i \in \Re^N$, $\boldsymbol{F}$ is a smooth function in its variables, $\overline{p}$ for *System A* is a fixed parameter value and p for *System B* is a externally adjustable parameter whose value is restrict to lie in some small interval,

$$|p - \overline{p}| < d_{lm}, \tag{14}$$

about $\overline{p}$, where d_{lm} is a small number defining the range of parameter variation. Suppose that the two systems start with different initial conditions. The resulting chaotic trajectories are completely uncorrelated. Some time later, due to ergodicity, the two trajectories can get arbitrarily close to each other. Consider that this event take places at the time *i*. The difference between the trajectories in the next iteration can be calculated using the following equation:

$$\boldsymbol{X}_{i+1} - \boldsymbol{Y}_{i+1} = \boldsymbol{F}(\boldsymbol{X}_i, p) - \boldsymbol{F}(\boldsymbol{Y}_i, \overline{p}). \tag{15}$$

For values of p close to $\overline{p}$, and as $\boldsymbol{X}_i$ falls in a small neighborhood of $\boldsymbol{Y}_i$, the previous equation can be approximated in the neighborhood of $\boldsymbol{Y}_i$ by the linear map

$$\boldsymbol{X}_{i+1} - \boldsymbol{Y}_{i+1}^{\overline{p}} = \boldsymbol{A}_i\left[\boldsymbol{X}_i - \boldsymbol{Y}_i^{\overline{p}}\right] + \boldsymbol{B}_i\left(p_i - \overline{p}\right), \tag{16}$$

where $\boldsymbol{A}_i$ is an $n \times n$ Jacobian matrix and $\boldsymbol{B}_i$ is an n-dimensional column vector,

$$\boldsymbol{A}_i = \boldsymbol{D}_{\boldsymbol{Z}}\boldsymbol{F}(\boldsymbol{Z}, p)|_{\boldsymbol{Z}=\boldsymbol{Y}_i^{\overline{p}},\, p=\overline{p}}, \tag{17}$$

$$\boldsymbol{B_i} = \boldsymbol{D}_p \boldsymbol{F}(\boldsymbol{Z}, p)|_{\boldsymbol{Z}=\boldsymbol{Y}_i^{\bar{p}}, p=\bar{p}}, \tag{18}$$

and these partial derivatives are evaluated at $\boldsymbol{Z} = \boldsymbol{Y}_i^{\bar{p}}$ and $p = \bar{p}$. Consider that the point of the trajectory of *System A*, u times ahead, has u unstable directions, and s stable directions in its tangent space. We can determine vectors $\{\boldsymbol{v}_{\boldsymbol{i+u,1}}, \boldsymbol{v}_{i+u,2}, \cdots, \boldsymbol{v}_{\boldsymbol{i+u,s}}\}$ which span the linearized stable subspace of the *System A* trajectory at the point $\boldsymbol{Y}_{i+u}^{\bar{p}}$. Let us define the matrix

$$\Phi_{i,j} = \boldsymbol{A}_{\boldsymbol{i+u-1}} \boldsymbol{A}_{\boldsymbol{i+u-2}} \cdots \boldsymbol{A}_{i+j+1} \boldsymbol{A}_{i+j}, \tag{19}$$

for $j = 1, 2, \cdots, (u-1)$. To derive the control action to be applied to the parameter p of system B at each iteration, we iterate Eq. (15) u times,

$$\boldsymbol{X}_{i+u} - \boldsymbol{Y}_{\boldsymbol{i+u}}^{\bar{p}} = \Phi_{i,0}\left[\boldsymbol{X}_i - \boldsymbol{Y}_{\boldsymbol{i}}^{\bar{p}}\right] + \Phi_{\boldsymbol{i,1}} \boldsymbol{B}_i (p_i - \bar{p}) + \cdots + \boldsymbol{B}_{i+u-1}(p_{i+u-1} - \bar{p}). \tag{20}$$

To make *System B* synchronize with *System A*, $\boldsymbol{X}_{\boldsymbol{i+u}}$ must land on the linearized stable manifold of *System A* trajectory at the point $\boldsymbol{Y}_{\boldsymbol{i+u}}^{\bar{p}}$. Thus, the parameter ps must be chosen such that there exists s coefficients $a_1, a_2, \cdots, a_s$ such that

$$\boldsymbol{X}_{\boldsymbol{i+u}} - \boldsymbol{Y}_{\boldsymbol{i+u}}^{\bar{p}} = a_1 \boldsymbol{v}_{i+u,1} + a_2 \boldsymbol{v}_{\boldsymbol{i+u,2}} + \cdots + a_s \boldsymbol{v}_{i+u,s}. \tag{21}$$

Considering Eqs. (20) and (21), we have a system of $u+s$ equations in $u+s$ unknowns variables $p_i, p_{i+1}, \cdots, p_{i+u-1}, a_1, a_2, \cdots, a_s$. This system can be solved for p_i to obtain

$$p_i - \bar{p} = -\boldsymbol{K}_i^t\left[\boldsymbol{X}_i - \boldsymbol{Y}_{\boldsymbol{i}}^{\bar{p}}\right], \tag{22}$$

where $\boldsymbol{K}_{\boldsymbol{i}}^{\boldsymbol{t}} = \boldsymbol{L}\boldsymbol{C}_i^{-1}\Phi_{i,0}$, $\boldsymbol{L} = [-1 \quad 0 \quad 0 \quad \cdots \quad 0]$, and $\boldsymbol{C}_i = [-\Phi_{\boldsymbol{i,1}}\boldsymbol{B}_i \quad \cdots \quad -\boldsymbol{B}_{i+u-1} \quad \boldsymbol{v}_{i+u,1} \quad \cdots \quad \boldsymbol{v}_{i+u,s}]$. When this method is applied, we find the values of the parameter p that must be applied to *System B* at each iteration for the synchronization of the two systems, *i.e.*, $\left|\boldsymbol{X}_{\boldsymbol{j}} - \boldsymbol{Y}_j^{\bar{p}}\right| \to 0$ for $j > i$.

In the previous equation, the state variables of the *System A* are estimated by using the Extended Kalman Filter (Brown and Hwang, 1997; Macau et al.,2002).

RESULTS

Let us now verify how our proposed method works in a physical meaning situation. Let us consider the double rotor map previously described. For $\boldsymbol{L}$ and $\boldsymbol{M}$ matrices, we use the same values that was used by Romeiras (Romeiras et al., 1992). Depending on the values of the parameters $c_1 = c_2 = f$, we face different dynamical behaviors for the system. Thus, in Figs. 2 and 3 we show the results of the use of our strategy to synchronize two double-rotor systems having parameters values of f equal to 8.0 and 9.0, respectively. For both situations the transmitted signal used is $z_i = H(\boldsymbol{X}_{\boldsymbol{i}}, v_i) = x(1) + x(2) + x(3) + x(4) + v$, where v is a white noise with normal probability distribution, *i. e.*, $p(v) \sim N(0, R)$ and R so that the signal to noise relation is equal to 70*dB*. For the first case, we are inside the region that occurs the transition from one positive Lyapunov exponent to two positive ones, while the finite-time Lyapunov exponents show fluctuations (Dawson et al., 1994) between one and two positive exponents. It means that we are in the region that appears the phenomenon of *unstable dimension variability*.

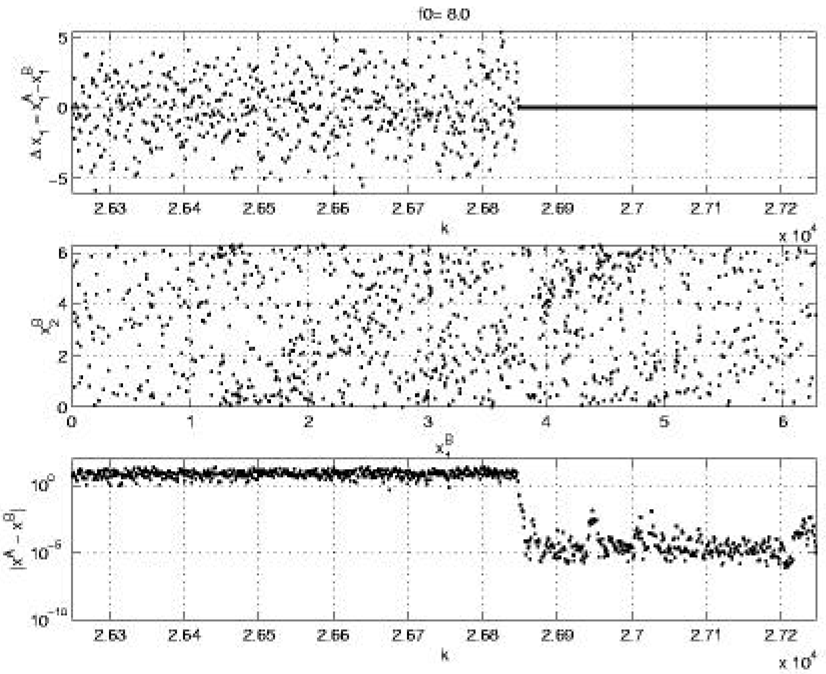

Fig2: Results of the use of our strategy to synchronize a double-rotor system having parameter value of *f=8.0*

For $f = 9.0$, the double-rotor is a hyperchaotic system, *i.e.*, the system has two positive Lyapunov exponent..

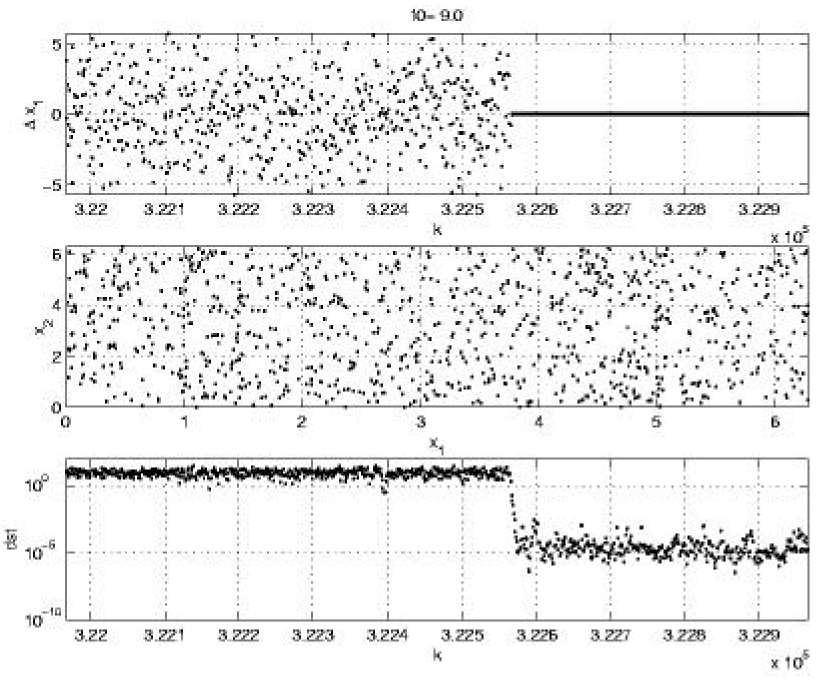

Fig 3: Results of the use of our strategy to synchronize a double-rotor system having parameter value of *f=9.0*

Comparing both results, we can verify that even in the presence of noise, as our synchronization algorithm is applied on the *system B*, its trajectory approaches the trajectory of *system A* and then both trajectories remain synchronized. Note, also, that the elapse of time until the trajectories come near to each other so that our active synchronization strategy can be applied is larger for the hyperchaotic regime than for the situation with *UDV*. Furthermore, form the results, we can verify that the synchronization is robust even in the presence of noise. Thus, our active synchronization method based on chaos control strategy and on the Extended Kalman Filter is capable of keep the systems synchronized as the trajectory evolves through the chaotic invariant set because it has a built-in adjustment mechanism that monitors the system local dynamics and adjust its coefficient according to the local dynamics.

Bibliography

(Abraham and Smale, 1970) R. Abraham and S. Smale, Proc. Symnpos. Pure Math., **14**, 5, Amer. Math. Soc., Providence, RI (1970).

(Afraimovich et al, 1986) V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, *Radio Phys. and Quantum Electronics*, **29**, 747 (1986).

(Ali and Fang, 1997) M. K. Ali and J. Q. Fang, *Phys. Rev. E*, **55**, 5285 (1997).

(Brown and Hwang, 1997) R. G. Brown and P. Y. C. Hwang, *Introduction to Random Signals and Applied Kalman Filtering*, 3d ed., John Wiley, New York (1997).

(Chua et al, 1993) L. º chua, M. Itoh, L. Kocarev, and K. Eckert, *J. circuits, Systems, and Computers* **3**, 93 (1993).

(Cuomo and Oppenheim, 1992) K. M. Cuomo and A. V. Oppenheim, *Phys. Rev. Lett.* **71**, 65 (1992).

(Cuomo et al, 1993) K. M. Cuomo, A. V. Oppenheim, and S. H. Strogatz, *Int. J. Bifurcation and Chaos* **3**, 1629 (1993).

(Dawson et al, 1994) S. P. Dawson, C. Grebogi, T. Sauer, and J. A Yorke, *Phys. Rev. Lett.* **73**, 1927 (1994).

(Dawson, 1996) S. P. Dawson, *Phys. Ver. Lett.* **76**, 4348 (1996).

(Devaney, 1989) R. Devaney, *An Introduction to Chaotic Dynamical Systems*, 2nd ed., Perseus Publishing (1989).

(Fink et al, 2000) K. S. Fink, G. Johnson, T. Carroll, D. Mar, and L. Pecora, *Phys. Rev. E* **61**, 5080 (2000).

(Fujisaka and Yamada, 1983) H. Fujisaka and T. Yamada, *Prog. Theor. Phys.* **69**, 32 (1983).

(Grebogi et al, 1990) C. Grebogi, S. Hammel, J. Yorke, and T. Sauer, *Phys. Rev. Lett.* **65**, 1527 (1990).

(Grebogi and Lai, 1997) C. Grebogi and Y. C. Lai, *Systems and Control Lett.* **31**, 407 (1997).

(Hammel et al, 1987) S. M. Hammel, J. A. Yorke, and C. Grebogi, *J. Complexity* **3**, 136 (1987).

(Heagy et al, 1994) J. F. Heagy, T. L. Carroll, and L. M. Pecora, *Phys. Rev. E* **50**, 1874 (1994).

(Johnson et al, 1998) G. ª Johnson, D. J. Mar., T. L. Carrol, and L. M. Pecora, *Phys. Rev. Lett.* **80**, 3956 (1998).

(Kalman, 1960) R. E. Kalman, *Trans. ASME – Journ. Basic Engineering* **D 83**, 35 (1960).

(Kocarev and Parlitz, 1996) L. Kocarev and U. Parlitz, *Phys. Rev. Lett.* **76**, 1816 (1996).

(Kostelich et al, 1997) E. J. Kostelich, I. Kan, C. Grebogi, E. Ott, and J. A. Yorke, *Physica D* **109**, 81 (1997).

(Lai et al, 1999) Y. C. Lai, C. Grebogi and J. Kurths, *Phys. Rev. E* **59**, 2907 (1999).

(Macau and Grebogi, 2000)

(Macau et al, 2002)

(Ott et al, 1990) E. Ott, C. Greboi and J. A. Yorke, *Phys. Rev. Lett.* **64**, 1196 (1990)

(Parlitz et al, 1992) U. Parlitz, L. º Chua, L. Kocarev, K. S. Halle and A. Shang, *Int. J. Bifurcation and Chaos* **2**, 973 (1992).

(Pecora and Carroll, 1990) L. M. Pecora and T. L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990).

(Pecora and Carroll, 1991) L. M. Pecora and T. L. Carroll, *Phys. Rev. A* **44**, 2374 (1991).

(Pecora et al, 1997) L. M. Pecora, T. L. Carroll, G. A. Johnson, D. J. Mar, and J. F. Heagy, *Chaos* **7**, 520 (1997).

(Peng et al, 1996) J. H. Peng, E. J. Ding, M. Ding, and W. Yang, *Phys. Rev. Lett.* **76**, 904 (1996).

(Pikovsky et al, 1997) A. Pikovsky, G. Osipov, M. Rosenbaum, M. Zaks, J. Kurths, *Phys. Rev. Lett.* **79**, 47 (1997).

(Pyragas, 1993) K. Pyragas, *Phys. Lett. A* **71**, 155 (1979).

(Romeiras et al, 1992) F. J. Romeiras, C. Grebogi, E. Ott and W. P. Dayawansa, *Physica D* **58**, 165 (1992).

(Sauer and Yorke, 1991) T. Sauer and J. A. Yorke, *Nonlinearity* **4**, 961 (1991).

(Sauer et al, 1997) T. Sauer, C. Grebogi and J. A. Yorke, *Phys. Rev. Lett.* **79**, 59 (1997).

(Tamasevicius and Cenys, 1997) A. Tamasevicius and A. Cenys, *Phys. Rev. E*, **55**, 297 (1997).