# Determining Stiffness Matrix by the Adjoint Method

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### Summary

The present investigation is focused on the solution of a dynamic inverse problem for damage identification in structures from measured data. The inverse problem is formulated as an optimization problem. It is solved using the Alifanov's variational approach, adapted by frequency domain measurements. The damage estimation has been evaluated using noiseless and noisy synthetic experimental data, considering a simple spring-mass system.

## Introduction

The direct solution of forced vibration problems are concerned with the determination of the system displacement, velocity and acceleration at time $t$ when the initial conditions, external forces, and time-dependent system parameters, such as stiffness coefficients and damping coefficients, are specified. On the other hand, the inverse vibration problems are concerned with the estimation of such quantities (stiffness or damping coefficients, external forces) from the measured vibration data, i.e., natural frequency and/or mode shape, or displacement measurements. One technique employed for solving inverse probems is the conjugate gradient method with the adjoint equation [Alifanov, 1974], [Chiwiacowsky and Campos-Velho, 2002], [Jarny et al., 1991], and a regularized solution through the genetic algorithm method [Chiwiacowsky and Campos-Velho, 2002], [Chiwiacowsky et al., 2003b].

The structural damage detection is displayed as an inverse vibration problem, since the damage evaluation is achieved through the determination of the stiffness coefficient variation, or the stiffness coefficient by itself. Recently this type of problem has already been solved employing the Alifanov's approach, i.e. the variational technique, where the results have been reported concerning lumped-parameter systems with a small number of degrees-of-freedom (DOFs) [Huang, 2002] or with a higher number of DOFs [Chiwiacowsky et al., 2003a] . Also, in some works more realistic structures have been considered such as truss and beam-like structures [Chiwiacowsky and Gasbarri, 2003], [Chiwiacowsky et al., 2004], [Castello and Rochina, 2002]. In the cited works the time-history of the displacement data have been adopted as the available experimental data.

In this work, the adjoint method has been applied to the inverse vibration problem of damage assessment in a lumped-parameter system with $N$-DOFs, assuming natural frequencies measurements as the available experimental data.

## The Inverse Analysis

The goal is to recovered the unknown stiffness coefficients from the synthetic system frequency measurements of a spring-mass system with $N$-DOF. The inverse anal-

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ysis with the conjugate gradient method involves the following steps [Alifanov, 1974], [Jarny et al., 1991]: (i) the solution of direct problem; (ii) the solution of sensitivity problem; (iii) the solution of adjoint problem and the gradient equation; (iv) the conjugate gradient method of minimization; (v) the stopping criteria. Next, a brief description of basic procedures involved in each of these steps is presented.

**The Direct Problem**

The undamped free vibration of a $N$-DOF structural system gives rise to the matrix eigenvalue problem,

$$\mathbf{K}\phi = \lambda \mathbf{M}\phi \,, \tag{1}$$

which will be considered as the direct eigenproblem in the frequency domain; being $\mathbf{K}$ and $\mathbf{M}$ the stiffness and mass matrices, $\lambda$ is the eigenvalue (natural frequency squared), and $\phi$ is the eigenvector.

**The Sensitivity Problem**

Since the problem involves $N$ unknown time-dependent stiffness parameters, which constitute the elements of the stiffness matrix $\mathbf{K} = f[\mathcal{K}]$, where $\mathcal{K} = [K_1, ..., K_N]$, in order to derive the sensitivity problem for each unknown function $K_i$, each unknown stiffness coefficient should be perturbed at a time. Supposing that the $K_i$ is perturbed by a small amount $\triangle K_{ij}\,\delta(i-j)$, where the $\delta(\cdot)$ is the Dirac-delta function and $j = 1, \ldots, N$, it results in a small change in frequencies and mode-shapes by the amounts $\triangle\lambda_{ij}(t)$ and $\triangle\phi_{ij}(t)$, respectively. Then, the sensitivity problem is obtained by replacing in the eigenvalue problem $K_i$ by $K_i + \triangle K_{ij}\,\delta(i-j)$, $\lambda_i$ by $\lambda_i + \triangle\lambda_{ij}$, $\phi_i$ by $\phi_i + \triangle\phi_{ij}$, and is given by

$$(\mathbf{K} - \lambda\mathbf{M})\,\phi = \mathbf{0} \;\Rightarrow\; [(\mathbf{K}_i + \triangle\mathbf{K}_j) - (\lambda_i + \triangle\lambda_{ij})\mathbf{M}]\,(\phi_i + \triangle\phi_{ij}) = \mathbf{0} \,. \tag{2}$$

Rearranging the terms of the above equation and subtracting from the resulting expression the original eigenvalue problem (1), after some algebraic manipulations and neglecting the second-order terms, the following expression yields

$$\phi_i^{\mathrm{T}}(\triangle\mathbf{K}_j - \triangle\lambda_{ij}\mathbf{M})\phi_i = \mathbf{0} \quad \Rightarrow \quad \triangle\lambda_{ij} = \frac{\phi_i^{\mathrm{T}}\,\triangle\mathbf{K}_j\,\phi_i}{\phi_i^{\mathrm{T}}\,\mathbf{M}\,\phi_i} \,, \tag{3}$$

which provides a sensibility analysis where the eigenvectors $\phi$ have been obtained when the updated stiffness matrix is considered.

**The Adjoint Problem and the Gradient Equation**

The inverse problem is to solved as an optimization problem, requiring that the unknown function $\mathcal{K}$ should minimize the functional $\mathbf{J}[\mathcal{K}]$ defined by

$$\mathbf{J}[\mathcal{K}] = \left\|\lambda^{exp} - \lambda\right\|_2^2 \,, \tag{4}$$

where $\lambda$ and $\lambda^{exp}$ are the computed and measured frequencies, respectively. For solving the minimization problem (4), the *Lagrange* multipliers $\psi$ are usually used to associate the constraints (1) to the functional form.

$$\mathcal{L}(\lambda, \mathcal{K}, \psi) = \left\| \lambda^{exp} - \lambda \right\|^2 - \psi^{\mathrm{T}} \left( \mathbf{K} - \lambda \mathbf{M} \right) \phi \ . \tag{5}$$

Knowing that the choice of the *Lagrange* multiplier is free, for convenience, it has been choose to be the solution of the adjoint problem:

$$\psi^{\mathrm{T}} \mathbf{M} \, \phi = -2(\lambda^{exp} - \lambda) \ . \tag{6}$$

Applying the variational theory [Woodbury, 2002], the left term is employed to determine the gradient $\mathbf{J}'[\mathcal{K}]$, which is given by

$$\mathbf{J}'[\mathcal{K}] = -\psi^{\mathrm{T}} \triangle \widetilde{\mathbf{K}} \, \phi \ , \tag{7}$$

where $\triangle \tilde{\mathbf{K}}_j$ refers to the $j$th perturbed stiffness matrix, i.e. $\triangle \tilde{\mathbf{K}}_j = \partial [\triangle \mathbf{K}] / \partial K_j$.

**The conjugate gradient method and the Stopping Criteria**

The iterative procedure based on the conjugate gradient method is used for the estimation of the unknown stiffness coefficients $\mathcal{K}$ given in the form [Jarny et al., 1991]:

$$\mathcal{K}^{n+1} = \mathcal{K}^n + \boldsymbol{\beta}^n \mathbf{P}^n, \quad n = 0, 1, 2, ...., \tag{8}$$

$$\mathbf{P}^n = \mathbf{J}'^n(t) + \boldsymbol{\gamma}^n \mathbf{P}^{n-1}, \quad \text{with} \quad \boldsymbol{\gamma}^0 = 0, \tag{9}$$

where $\boldsymbol{\beta}^n$ is the step size vector, $\mathbf{P}^n$ is the direction of descent vector and $\boldsymbol{\gamma}^n$ is the conjugate coefficient vector. The step size vector $\boldsymbol{\beta}^n$, appearing in Eq. (8), is determined by minimizing the functional vector $\mathbf{J}[\mathcal{K}^{n+1}]$ given by Eq. (4) with respect $\boldsymbol{\beta}^n$. The discrepancy principle for the stopping criteria is taken as

$$\mathbf{J}[\mathcal{K}^{n+1}] < \varepsilon^2. \tag{10}$$

where $\varepsilon^2 = \| \ \sigma \ \|^2$, and $\sigma$ is the standard deviation of the measurements errors.

## The Solution Algorithm

The procedure for the adjoint method can be summarized as:

**Step 1:** Choose an initial guess $\mathcal{K}^0$ – for example, $\mathcal{K}^0 =$ constant.

**Step 2:** Solve the direct eigenvalue problem [Eq. (1)], to obtain $\lambda$.

**Step 3:** Solve the adjoint problem [Eq. (6)], to obtain the *Lagrange* multiplier vector $\psi$.

**Step 4:** Knowing $\psi$, compute the gradient function vector $\mathbf{J}'(\mathcal{K})$ from Eq. (7).

**Step 5:** Compute the conjugate coefficient vector $\boldsymbol{\gamma}^n$.

**Step 6:** Compute the direction of descent vector $\mathbf{P}^n$ from Eq. (9).

**Step 7:** Setting $\triangle\mathbf{K} = \mathbf{P}^n$, solve the sensitivity problem [Eq. (3)], to obtain $\triangle\lambda$.

**Step 8:** Compute the step size $\boldsymbol{\beta}^n$.

**Step 9:** Compute $\mathcal{K}^{n+1}$ from Eq. (8).

**Step 10:** Test if the stopping criteria, Eq. (10), is satisfied. If not, go to step 2.

## Results and Dicussion

The parameters which define the undamaged configuration of the spring-mass are taken as: $M_i = 10.0$ kg, and $K_i = 2 \times 10^5$ N/m, where $i = 1,\ldots,10$. The following damage configuration has been considered: a 10% damage over the element 1; 25% over the element 3; 15% over the element 4; 5% over the element 5; 30% over the element 7; 20% over the element 8 and a 10% damage over the element 10. All the others elements have been assumed as undamaged for the generation of the experimental data.

The experimental data (*measured frequencies*) have been obtained from the exact solution of the eigenvalue problem (noiseless data) and then adding a random perturbation (noisy data),

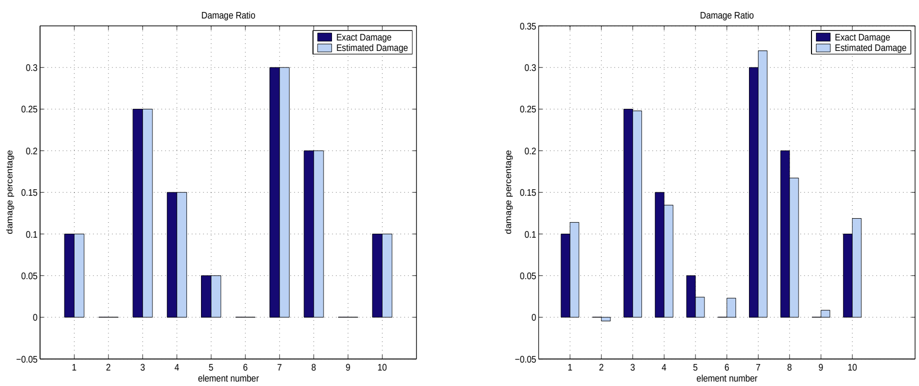
$$\lambda^{exp} = \lambda + \sigma\mathcal{R}, \qquad (11)$$

where $\mathcal{R}$ is a random variable from a normal distribution $\mathcal{R} \sim \mathcal{N}$ormal(0;1). It has been adopted $\sigma = 0.01$ . Also, the comparison between the estimated and exact values has been done through the use of the damage factor, defined as

$$DF_i = \frac{K_i^u - K_i^d}{K_i^u} \quad i = 1,\ldots,N \qquad (12)$$

where $K_i^u$ and $K_i^d$ are the undamaged and damaged parameters, respectively.

The numerical results have been obtained considering no prior information about the functional form of the unknown quantities, however undamaged stiffness values have been assumed as available, taken as initial guess (**Step 1**). Figure 1 shows estimated and exact damage factor values, for (a) noiseless and (b) noisy experimental data, respectively. Good estimations are obtained even for noise data.




(a) (b)

Figure 1: Estimated damage factor: (a) noiseless data, (b) noisy data.

**Final Remarks**

The evaluation of the conjugate gradient method with the adjoint equation on the estimation of stiffness coefficients (damage identification) has been considered. A simple dynamical system has been adopted to verify the feasibility of the Alifanov's method considering damage scenarios and employing synthetic noiseless and noisy frequencies measurements. Perfect reconstructions have been achieved for the noiseless data, and satisfactory estimations for the noisy data.

Futute works include more realistic structures, as well as other inverse vibration problems, such as damping matrix identification.

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References

[Alifanov, 1974] Alifanov, O. M. (1974). Solution of an inverse problem of heat conduction by iteration methods. *J. Eng. Phys.*, 26(4):471–476.

[Castello and Rochinha, 2002] Castello, D. A. and Rochinha, F. A. (2002). On the identification of constitutive parameters of viscoelastic materials by means of a time domain technique. Angra dos Reis (RJ), Brazil. Proceedings of the 4th International Conference on Inverse Problems in Engineering: Theory and Practice.

[Chiwiacowsky and Campos-Velho, 2002] Chiwiacowsky, L. D. and Campos-Velho, H. F. (2002). Different approaches for the solution of a backward heat conduction problem. *Inverse Probl. Eng.*, 11(3):471–494.

[Chiwiacowsky et al., 2003a] Chiwiacowsky, L. D., Campos-Velho, H. F., and Gasbarri, P. (2003a). The damage identification problem: a hybrid approach. São José dos Campos (SP), Brazil. Proceedings of the 2nd Thematic Congress on Dynamics and Control pages 1393-1402, ISBN 85-86883-15-8.

[Chiwiacowsky et al., 2004] Chiwiacowsky, L. D., Campos-Velho, H. F., and Gasbarri, P. (2004). A variational approach for solving an inverse vibration problem. Rio de Janeiro (RJ), Brazil. Proceedings of the Inverse Problems, Design and Optimization Symposium, paper code IPDO-078.

[Chiwiacowsky et al., 2003b] Chiwiacowsky, L. D., Campos-Velho, H. F., Preto, A. J., and Stephany, S. (2003b). Identifying initial condition in heat conduction transfer by a genetic algorithm: A parallel approach. Ouro Preto (MG), Brazil. Proceedings of the XXIV Iberian Latin-American Congress on Computational Methods in Engineering.

[Chiwiacowsky and Gasbarri, 2003] Chiwiacowsky, L. D. and Gasbarri, P. (2003). he damage identification problem: a hybrid approach. Rome, Italy. Proceedings of the XVII AIDAA, Vol. 2, pages 1403-1414.

[Huang, 2002] Huang, C. H. (2002). An inverse vibration problem for simultaneously estimating the time-dependent stiffness coefficients. Angra dos Reis (RJ), Brazil. Proceedings of the 4th International Conference on Inverse Problems in Engineering: Theory and Practice.

[Jarny et al., 1991] Jarny, Y., Özisik, M. N., and Bardon, J. P. (1991). A general optimization method using adjoint equation for solving multidimensional inverse heat conduction. *Int. J. Heat Mass Tran.*, 34(11):2911–2919.

[Woodbury, 2002] Woodbury, K. A. (2002). *Inverse Engineering Handbook*. CRC Press.