MINISTÉRIO DA CIÊNCIA E TECNOLOGIA
INSTITUTO NACIONAL DE PESQUISAS ESPACIAIS

INPE-11301-PRE/6738

NONIMPULSIVE ORBITAL MANEUVERS UNDER THRUST DEVIATIONS EFFECT: CAUSE/EFFECT ALGEBRAIC RELATION TO SEMI-MAJOR AXIS

Antônio Delson C. de Jesus*
Marcelo Lopes Oliveira e Souza
Antônio Fernando Bertachini de Almeida Prado

ADVANCES IN SPACE DYNAMICS 4: CELESTIAL MECHANICS AND ASTRONAUTICS,
H. K. Kuga, Editor, 221-232 (2004).
Instituto Nacional de Pesquisas Espaciais – INPE, São José dos Campos, SP, Brazil.
ISBN 85-17-00012-9

INPE
São José dos Campos
2004

ADVANCES IN SPACE DYNAMICS 4: CELESTIAL MECHANICS AND ASTRONAUTICS,
H. K. Kuga, Editor, 221-232 (2004).
Instituto Nacional de Pesquisas Espaciais – INPE, São José dos Campos, SP, Brazil.
ISBN 85-17-00012-9

NONIMPULSIVE ORBITAL MANEUVERS UNDER THRUST DEVIATIONS EFFECT: CAUSE/EFFECT ALGEBRAIC RELATION TO SEMI-MAJOR AXIS

Antônio Delson C. de Jesus*
Marcelo Lopes Oliveira e Souza
Antônio Fernando Bertachini de Almeida Prado
Departamento de Física da Universidade Estadual de Feira de Santana
Km 03, BR 116, Campus Universitário, 44.031-460, Feira de Santana/BA
**E-mail: adj@uefs.br*

ABSTRACT

In this paper we present the cause/effect relation between the thrust vector errors and the final orbit semi-major axis to the spatial vehicles orbital transfer, under non linear and stochastic keplerian dynamic with optimum fuel consumption. We found the algebraic relations which represent the dynamic effect along the transfer maneuver and the final trajectory, through the semi-major axis and the deviations produced in the thrust vector. These results confirmed the JSP1 curves family, found before in the Monte-Carlo analysis, which showed the loss of optimality and the order precision in the vehicles final trajectory. The relations showed these effects in the final orbit are proportional to the even power of the causes deviations.

INTRODUCTION

The nonlinear keplerian dynamic introduces difficult for the analytical solution of the actual orbital transfers. Only the Two Body Problem has analytical solution for the gravitational force between the bodies. The transfers maneuvers involves others forces effects, particularly, the thrust vector to the propulsion system. So, we do not have the analytical solution for the general and actual orbital transfers problem. Besides these difficult, associate to the non-ideal propulsion system (actual) there are many deviations causes. The mathematical treatment exact in these cases is not efficient. In general, the differential equations are simplified and the solution loss the important information about the fenomenum studied. This paper presents results algebraic about the orbital transfers under thrust deviations. Most space missions need trajectory/orbit transfers and they have linear and/or angular misalignments that displace the vehicle with respect its nominal directions. The mathematical treatment for these deviations can be realized under many approaches. In the deterministic approach we can review Schwend and Strobl (1977), Tandon (1988), Rodrigues (1991), Santos-Paulo (1998), Rocco (1999) and Schultz (1997), among others.

In the probabilistic approach Porcelli and Vogel (1980) presented an algorithm for the determination of the orbit insertion errors in biimpulsive noncoplanar orbital transfers (perigee and appogee), using the covariance matrices of the sources of errors. Adams and Melton (1986) extended such algorithm to ascent transfers under a finite thrust, modeled as a sequence of impulsive burns. They developed an algorithm to compute the propagation of the navigation and direction errors among the nominal trajectory, with finite perigee burns. Rao (1993) built a semi-analytic theory to extend covariance analysis to long-term errors on elliptical orbits. Howell and Gordon (1994) also applied covariance analysis to the orbit determination errors and they develop a station-keeping strategy of Sun-Earth L1 libration point orbits. Junkins (1996) et alli and Junkins (1997) discussed the precision of the error covariance matrix method through nonlinear transformations of coordinates. He also found a progressive deformation of the initial ellipsoid of trajectory distribution (due to gaussian initial condition errors), that was not anticipated by the covariance analysis of linearized models with zero mean errors. Carlton-Wippern (1997) proposed differential equations in polar coordinates for the growth of the mean position errors of satellites (due to errors in the initial conditions or in the drag), by using an approximation of Langevin's equation and a first order perturbation theory. Alfriend (1999) studied the effects of drag uncertainty via covariance analysis. In the mimimax approach the russian authors are mainly.

However, all these analyses are approximated. This motivated an exhaustive numerical (Jesus, 2000a,b) but exact analysis (by Monte-Carlo), and a partial algebraic analysis done by Jesus (1999). In this work we present the first part of the algebraic analysis of nonimpulsive orbital transfers under thrust errors. The results were obtained for two transfers: the first, a low thrust transfer between high coplanar orbits (we call it "theoretical transfer"), used by Biggs (1978,1979) and Prado (1989); the second, a high thrust transfer between middle noncoplanar orbits (the first transfer of the EUTELSAT II-F2 satellite, we call it "practical transfer") implemented by Kuga (1991) et alli. The simulations were done for both transfers with minimum fuel consumption. The "pitch" and "yaw" angles were taken as control variables such that the overall minimum fuel consumption defines each burn of the thrusters. The errors sources that we considered were the magnitude errors, "pitch" and "yaw" directions errors of the thrust vector, as causes of the deviations found in the several keplerian elements of the transfer trajectory. These error sources (random-bias and white-noise errors) introduced in the orbital transfer dynamics produce effects in the final orbit keplerian elements in the final instant.

In this work we present an algebraic analysis of the effects of these errors on the mean of the deviations of the keplerian elements of the final orbit with respect to the reference orbit (final orbit without errors in the thrust vector) for both transfers. The approach that we used in this work for the treatment of the errors was the probabilistic one, assuming these as having zero mean unit variance gaussian probability density function and having zero mean uniform probability density function.

NONIMPULSIVE AND COPLANAR ORBITAL MANEUVERS – ALGEBRAIC ANALYSIS

The orbital transfer problem studied was formulated for the minimum fuel consumption with respect to thrust direction, that is, w.r.t. the “pitch” (α) and “yaw” (β) angles, subjected to the dynamics in inertial coordinates, helped by the instantaneous keplerian coordinates (Ω,I,ω,f, a, e) and rewriting by the other coordinate system centered in the satellite and the 9 state variables, defined and used by Biggs(1978,1979) and Prado (1989). The geometric development of these coordinates systems can be found in Jesus (2002). In the centered-satellite-system we decomposed the actual thrust vector in three components, radial, transversal and normal directions.

The orbital transfers can be economic or spent. Those more economics orbits are of the practical interest and the most them occur in-plane, that is, with the "yaw" angle null (β=0). That is, the general and preferential missions are those without plane change. The numerical results found by Jesus (1999) showed too dependence of the semi-major axis deviations with “yaw” deviations. In this paper, we choose for our algebraic analysis the in-plane (α ≠0 and β=0) transfer maneuvers. We choose too, F and m constants.

We can write the move plane equations:

$$F_t = m.\dot{v}_t(t) = F.\cos\alpha(t) - m.v_r(t).\dot{f}(t) \tag{1}$$

$$F_r = m.\dot{v}_r(t) = F.\operatorname{sen}\alpha(t) - \frac{\mu.m}{r^2(t)} + m.v_t(t).\dot{f}(t) \tag{2}$$

$$\dot{f}(t) = \frac{v_t(t)}{r(t)} \tag{3}$$

$$\dot{r}(t) = v_r(t) \tag{4}$$

with,

F_t and F_r the transversal and radial components of the thrust vector, respectively;
$\dot{v}_t(t), \dot{v}_r(t)$ the transversal and radial components of the accelerations, respectively;
$v_t(t), v_r(t)$ the transversal and radial components of the velocities, respectively;
$\dot{f}(t)$ the angular velocity;
r(t) the vector position between satellite and central body.

If the transfer were noncoplanar (β≠0) the components thrust equations would be:

$$\vec{F}_E = \vec{F} + \Delta\vec{F} \tag{5}$$

$$\vec{F}_E = \vec{F}_R + \vec{F}_T + \vec{F}_N \tag{6}$$

$$\left|\vec{F}_E\right| = F_E, \quad \left|\vec{F}\right| = F \tag{7}$$

$$F_R = (F + \Delta F).\cos(\beta + \Delta\beta).\operatorname{sen}(\alpha + \Delta\alpha) \tag{8}$$

$$F_T = (F + \Delta F).\cos(\beta + \Delta\beta).\cos(\alpha + \Delta\alpha) \tag{9}$$

$$F_N = (F + \Delta F).\operatorname{sen}(\beta + \Delta\beta) \tag{10}$$

Our algebraic approach for the semi-major axis deviations is done through the rate variation of the satellite mechanic energy, which is equal the integral of the potencies changed with results forces components in the transversal and radial directions. The kinematics energy variations are:

$$\frac{d[E_c(t)]_r}{dt} = m.v_r(t).\dot{v}_r(t) \tag{11}$$

$$\frac{d[E_c(t)]_t}{dt} = m.v_t(t).\dot{v}_t(t) \tag{12}$$

$$\frac{\mathrm{dE_p(t)}}{\mathrm{dt}} = \frac{\mu.\mathrm{m.v_r(t)}}{\mathrm{r^2(t)}} \tag{13}$$

Add these equations we obtain the variation of the satellite mechanic energy, E_m, without "pitch" error,

$$\frac{dE_m(t)}{dt} = F.\cos\alpha(t).v_t(t) + F.\operatorname{sen}\alpha(t).v_r(t) \tag{14}$$

or, during the time interval Δt,

$$\Delta E_m(t_1,t_2) = E_m(t_2) - E_m(t_1) = \int_{t_1}^{t_2} F.\left(\cos\alpha(t).v_t(t) + \operatorname{sen}\alpha(t).v_r(t)\right).dt$$

$$= \frac{-\mu.m}{2.a(t_2)} + \frac{\mu.m}{2.a(t_1)} \tag{15}$$

with,

$a(t_i)$ the semi-major axis of the satellite orbit of the instant i.

The equation (15) for one transfer under "pitch" error, $\Delta\alpha(t)$ is,

$$\Delta E'_m(t_1,t_2) = E'_m(t_2) - E'_m(t_1) = \int_{t_1}^{t_2} F.\left(\cos[\alpha(t) + \Delta\alpha(t)].v'_t(t)\right).dt\ +$$

$$\int_{t_1}^{t_2} F.\left(\operatorname{sen}[\alpha(t) + \Delta\alpha(t)].v'_r(t)\right).dt \tag{16}$$

Taking the difference between the both equations, (15) and (16), we obtain,

$$\Delta_2 E_m(t_1,t_2) \equiv \Delta E'_m(t_1,t_2) - \Delta E_m(t_1,t_2) = \frac{-\mu.m}{2.a'(t_2)} + \frac{\mu.m}{2.a'(t_1)} +$$

$$\frac{\mu.m}{2.a^2(t_2)} - \frac{\mu.m}{2.a^2(t_1)} = \int_{t_1}^{t_2} F.\left(\cos[\alpha(t) + \Delta\alpha(t)].v'_t(t) - \cos\alpha(t).v_t(t)\right).dt +$$

$$\int_{t_1}^{t_2} F.\left(\operatorname{sen}[\alpha(t) + \Delta\alpha(t)].v'_r(t) - \operatorname{sen}\alpha(t).v_r(t)\right).dt \tag{17}$$

If we use the fact that the semi-major axis of the departure and arrive orbits in the initial instant are equal and doing after some algebraic manipulation, taking it the expectation or first moment, E, of the final equation, we have,

$$\mathrm{E}\{\Delta_2 E_m(t_1,t_2)\} = \mathrm{E}\{\int_{t_1}^{t_2} F.\{\cos\alpha(t).[\cos[\Delta\alpha(t)]-1]-\mathrm{sen}[\alpha(t)].\mathrm{sen}[\Delta\alpha(t)]\}.v_t^{'}(t).dt\}+$$

$$\mathrm{E}\{\int_{t_1}^{t_2} F.\{\mathrm{sen}\,\alpha(t).[\cos[\Delta\alpha(t)]-1]+\cos[\alpha(t)].\mathrm{sen}[\Delta\alpha(t)]\}.v_r^{'}(t).dt\}+$$

$$\mathrm{E}\{\int_{t_1}^{t_2} F.\cos[\Delta\alpha(t)].\{v_t^{'}(t)-v_t(t)\}.dt\}+\mathrm{E}\{\int_{t_1}^{t_2} F.\mathrm{sen}[\Delta\alpha(t)].\{v_r^{'}(t)-v_r(t)\}.dt\} \quad (18)$$

Now, we consider the stochastic process are ergotic. So, the expectation operator (mean inside the ensemble) commutes with the integral operator (in the time). Besides this, the function F and the trigonometric functions are deterministic in the time. Therefore, we evaluate the mean through the ensemble for the equation (18),

$$\mathrm{E}\{\Delta_2 E_m(t_1,t_2)\} = \int_{t_1}^{t_2} F.\cos\alpha(t).\mathrm{E}\{[\cos[\Delta\alpha(t)]-1].v_t^{'}(t)\}.dt -$$

$$\int_{t_1}^{t_2} F.\mathrm{sen}\,\alpha(t).\mathrm{E}\ \{\mathrm{sen}[\Delta\alpha(t)].v_t^{'}(t)\}.dt + \int_{t_1}^{t_2} F.\mathrm{sen}\,\alpha(t).\mathrm{E}\ \{[\cos[\Delta\alpha(t)]-1].v_r^{'}(t)\}.dt +$$

$$\int_{t_1}^{t_2} F.\cos\alpha(t).\mathrm{E}\ \{\mathrm{sen}[\Delta\alpha(t)].v_r^{'}(t)\}.dt + \int_{t_1}^{t_2} F.\cos[\alpha(t)].\mathrm{E}\ .\{v_t^{'}(t)-v_t(t)\}.\mathrm{dt}\ +$$

$$\int_{t_1}^{t_2} F.\mathrm{sen}[\alpha(t)].\mathrm{E}.\{v_r^{'}(t)-v_r(t)\}.\mathrm{dt} \quad (19)$$

The expression (19) is general for any probability distribution function to $\Delta\alpha(t)$ and for any kind of noise, that is, "white-noise", "pink-noise" or other. But, we must define if the variables inside the integral in (19) are correlated or not correlate to evaluate the expectation.

Case $\Delta\alpha(t)$ Not Correlated with Transversal and Radial Velocities, Uniform and Gaussian

In this case (white-noise), we decompose the expectation operator as one product of the individual expectations for the trigonometric functions of the $\Delta\alpha(t)$ and the velocities components, because they are not correlated. For the $\Delta\alpha(t)$ with uniform distribution inside the interval $[-\Delta\alpha_{máx}, \Delta\alpha_{máx}]$, we have,

$$\mathrm{E}\{[\cos[\Delta\alpha(t_1)]-1].v_t^{'}(t_1)\} = \mathrm{E}\{[\cos[\Delta\alpha(t_1)]-1]\}.\mathrm{E}\{v_t^{'}(t_1)\} =$$

$$v_t(t_1).\mathrm{E}\{[\cos[\Delta\alpha(t_1)]-1]\} = [\mathrm{E}\ \{[\cos[\Delta\alpha(t_1)]]\} \text{- E}\ \{1\}].\,v_t(t_1) =$$

$$v_t(t_1).\{\frac{1}{2.\Delta\alpha_{máx}}.\int_{-\Delta\alpha_{máx}}^{\Delta\alpha_{máx}}\cos[\Delta\alpha.]d(\Delta\alpha)-1\}=\{\frac{1}{2.\Delta\alpha_{máx}}.\mathrm{sen}[\Delta\alpha]\Big|_{-\Delta\alpha_{máx}}^{\Delta\alpha_{máx}}-1\}.v_t(t_1)=$$

$$v_t(t_1)\left\{\frac{\mathrm{sen}[\Delta\alpha_{máx}]}{\Delta\alpha_{máx}}-1\right\} \qquad \text{and} \tag{20}$$

$$\mathrm{E}\{[\cos[\Delta\alpha(t)]-1].v_r^{'}(t_1)\}=v_r(t_1)\left\{\frac{\mathrm{sen}[\Delta\alpha_{máx}]}{\Delta\alpha_{máx}}-1\right\} \tag{21}$$

$$\mathrm{E}\{\mathrm{sen}[\Delta\alpha(t_1)].v_t^{'}(t_1)\}=\mathrm{E}\{\mathrm{sen}[\Delta\alpha(t_1)]\}.\mathrm{E}\{v_t^{'}(t_1)\}=$$

$$v_t(t_1).\frac{1}{2.\Delta\alpha_{máx}}.\int_{-\Delta\alpha_{máx}}^{\Delta\alpha_{máx}}\mathrm{sen}[\Delta\alpha.]d(\Delta\alpha)=\frac{1}{2.\Delta\alpha_{máx}}.\cos[\Delta\alpha]\Big|_{-\Delta\alpha_{máx}}^{\Delta\alpha_{máx}}=0 \tag{22}$$

$$\mathrm{E}\{\mathrm{sen}[\Delta\alpha(t_1)].v_r^{'}(t_1)\}=0 \tag{23}$$

We consider that the velocities effects inside the interval [$-\Delta\alpha_{máx}$, $\Delta\alpha_{máx}$] in the same time are, practically, balanced, because the deviations occur between values maxima and minima inside them. That is, the velocities with errors and without them are very close values. So,

$$\mathrm{E}\{v_{t,r}^{'}(t)\}=v_{t,r}(t_1) \tag{24}$$

The equation (19) with this results is,

$$\mathrm{E}\{\Delta_2E_m(t_1,t_2)\}=\int_{t_1}^{t_2}F.\cos\alpha(t).v_t(t)\left\{\frac{\mathrm{sen}[\Delta\alpha_{máx}]}{\Delta\alpha_{máx}}-1\right\}.\mathrm{dt}+$$

$$\int_{t_1}^{t_2}F.\mathrm{sen}\,\alpha(t).v_r(t)\left\{\frac{\mathrm{sen}[\Delta\alpha_{máx}]}{\Delta\alpha_{máx}}-1\right\}.\mathrm{dt} \tag{25}$$

In other hand, we have,

$$\mathrm{E}\{\Delta_2E_m(t_1,t_2)\}=\mathrm{E}\{\frac{\mu.m}{2.a(t_2)}\frac{-\mu.m}{2.a^{'}(t_2)}\}=\frac{\mu.m}{2}.\frac{1}{a(t_2)}\mathrm{E}\{\frac{\Delta a(t_2)}{a^{'}(t_2)}\} \tag{26}$$

with,

$$\Delta a(t_2) = a'(t_2) - a(t) \tag{27}$$

If we expand the expression (26) in turn of the rate $\Delta a(t_2)/a(t_2)$,

$$\frac{\mu.m}{2}.[\frac{1}{a^2(t_2)}.\mathrm{E}\{\Delta a(t_2)\}-\frac{1}{a^3(t_2)}.\mathrm{E}\{\Delta^2 a(t_2)\}+\frac{1}{a^4(t_2)}.\mathrm{E}\{\Delta^3 a(t_2)\}$$

$$-\frac{1}{a^5(t_2)}.\mathrm{E}\{\Delta^4 a(t_2)\}+...] = \frac{\mu.m}{2}.\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}.\mathrm{E}\{\Delta^n a(t_2)\} \quad (28)$$

We can compare the both equations, (28) and (25),

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}.\mathrm{E}\ \{\Delta^n a(t_2)\} = \mathrm{K}_1\ .\left\{\frac{\mathrm{sen}[\Delta\alpha_{máx}]}{\Delta\alpha_{máx}}-1\right\}=$$

$$\mathrm{K}_1.\left\{-\frac{1}{3!}.\Delta^2\alpha_{máx}+\frac{1}{5!}.\Delta^4\alpha_{máx}-\frac{1}{7!}.\Delta^6\alpha_{máx}+...\right\} \text{ or}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}.\mathrm{E}\{\Delta^n a(t_2)\}= \mathrm{K}_1.\sum_{n=1}^{\infty}(-1)^n.\frac{1}{(2.n+1)!}.\Delta^{2.n}\alpha_{máx} \quad (29)$$

with,

$$\mathrm{K}_1 = \frac{2}{\mu.}.(\mathrm{Q}_1+\mathrm{Q}_2)/\mathrm{m} \quad (30)$$

Q_1 and Q_2 are quadratures.

The equation (29) describes one sequence of the even power terms for the maximum deviation in "pitch" with respect the expected values of the semi-major axis. For n=1, we have,

$$\mathrm{E}\{\Delta a(t_2)\}= -\frac{1}{3!}.\Delta^2\alpha_{máx}\ \mathrm{K}_1.\mathrm{a}^2(\mathrm{t}_2) = -\frac{1}{3!}.\Delta^2\alpha_{máx}\ \mathrm{K}_2 \quad (31)$$

with,

$$\mathrm{K}_2 = \mathrm{K}_1.\mathrm{a}^2(\mathrm{t}_2) \quad (32)$$

This result shows that for the first order the cause/effect relationship is one parabolic fitting. But, the general curve would be one composition of the all even power terms.

The procedures for the $\Delta\alpha(t)$ with guassian distribution inside the interval [$-\Delta\alpha_{máx}$, $\Delta\alpha_{máx}$] are the same of the uniform distribution. So, the

$$[\mathrm{E}\{[\cos[\Delta\alpha(t_1)]]\}-1].v_t(t_1)= v_t(t_1).\{\int_{-\infty}^{\infty}\cos[\Delta\alpha].\frac{e^{-\frac{(\Delta\alpha)^2}{2.\sigma_\alpha}}}{\sqrt{2.\pi}.\sigma_\alpha}.d(\Delta\alpha)\ -1\}=$$

$$v_t(t_1)\,.\{e^{-\frac{\sigma_\alpha^2}{2}}\text{-}1\}=\ v_t(t_1)\,.\{-\frac{\sigma_\alpha^2}{2}+\frac{\sigma_\alpha^4}{8}-\frac{\sigma_\alpha^6}{48}+...\}\qquad \text{or} \qquad (33)$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{1}{a^{n+1}(t_2)}\,.\mathrm{E}\{\,\Delta^n a(t_2)\,\}\,=\mathrm{K}_1.\sum_{n=1}^{\infty}\,(-1)^n\,.\frac{\sigma_\alpha^{2.n}}{2^n.n!} \qquad (34)$$

The form of the curve in (34) is similar that in (29). That is, there is one clear non-linear relationship between cause ($\Delta\alpha_{máx}=\sqrt{3}.\sigma_\alpha$) and effects ($\Delta a(t_2)$). For n=1, we have,

$$\mathrm{E}\{\,\Delta a(t_2)\,\}\,=-\frac{1}{6}.\sigma_\alpha^2.\mathrm{K}_2 \qquad (35)$$

Case Δα(t) Correlated with Transversal and Radial velocities, uniform and Gaussian

In this case (pink-noise), we cannot decompose the expectation operator as one product of the individual expectations for the trigonometric functions of the Δα(t) and the velocities components, because now they are correlated. The procedures are the same done until this point, except, we must evaluate the expectation of the products of the different variables, without the decompose them. Besides this, we consider the Δα(t) random-bias deviations, that is, Δα(t) = constant = $\Delta\alpha(t_1)$= Δα.

After mathematical manipulations we found the following equation, for the both cases, uniform and gaussian distribution,

$$\mathrm{I}_{r,t} = \int_{t_1}^{t_2} dt\ \mathrm{E}\ \left\{\cos[\Delta\alpha].\dot{v}_{r,t}^{'}(t).\dot{f}^{'}(t)\right\}$$

(36)

We know that the integral of the odd functions is null for the symmetrical distributions is null. But, (36) has one even product of the functions. The odd functions inside de product are not known, but we can modeled its product as one even function, for example, $\{\cos[\Delta\alpha]\}$. Other important approach in this way is to consider for the equation (2) that the expectation of the product is equal the product of the expectations of the functions, so that,

$$\mathrm{E}\left\{\frac{\cos[\Delta\alpha]}{r^{'2}(t)}\right\}=\mathrm{E}\left\{\cos[\Delta\alpha].\frac{1}{r^{'2}(t)}\right\}\cong\mathrm{E}\left\{\cos[\Delta\alpha]\right\}.\mathrm{E}\left\{1/r^{'2}(t)\right\}=\mathrm{E}\left\{\cos[\Delta\alpha]\right\}/\{r^2(t)\} \qquad (37)$$

The final forms are:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{1}{a^{n+1}(t_2)}\,.\mathrm{E}\{\,\Delta^n a(t_2)\,\}=\lambda_1-\ \lambda_2\,.\Delta^2\alpha_{máx}+\lambda_3\,.\,\Delta^4\alpha_{máx}-\ \ldots \qquad (38)$$

for the uniform case and,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{a^{n+1}(t_2)}.\mathrm{E}\{\Delta^n a(t_2)\} = \lambda_4 - \lambda_5 . \sigma_\alpha^2 + \lambda_6 . \sigma_\alpha^4 - \lambda_7 . \sigma_\alpha^6 + \quad ... \tag{39}$$

for the gaussian case, where the coefficients are

$$\lambda_1 = Q_8.(Q_5 + Q_6 - v_t(t_1)) + Q_{12}.(Q_{10} + Q_3 - v_r(t_1)) \tag{40}$$

$$\lambda_2 = \left((2.Q_3.Q_{12} - Q_8.Q_4 + Q_8.Q_5).\left(\frac{1}{2.2!}+\frac{1}{2.3!}\right) + (Q_{10}.Q_{12} + Q_8.Q_6 + Q_8.Q_5).\left(\frac{1}{3!}\right)\right) \tag{41}$$

$$\lambda_3 = \left((-Q_8.Q_4 + Q_8.Q_5).\left(\frac{1}{2.2!.3!}+\frac{1}{2.4!}\right) + \frac{1}{3!}.(Q_6 + Q_{10}.Q_{12}) + \frac{Q_8.Q_5}{7!} - \frac{Q_{12}.Q_3}{2.5!}\right) \tag{42}$$

$$\lambda_4 = Q_8.(Q_6 - v_t(t_1)) + Q_{12}.(Q_{11} - v_r(t_1)) \tag{43}$$

$$\lambda_5 = Q_{12} + \frac{Q_8.Q_6}{2} - Q_8.Q_4 + Q_{12}.Q_{11,1} \tag{44}$$

$$\lambda_6 = Q_{12} + \frac{Q_8.Q_6}{8} - Q_8.Q_4 + Q_{12}.Q_{11,2} \tag{45}$$

$$\lambda_7 = \frac{2}{3}.Q_{12} + \frac{Q_8.Q_6}{48} - \frac{2}{3}.Q_8.Q_4 + Q_{12}.Q_{11,3} \tag{46}$$

The Q_{ij} functions are quadratures. For n=1 we have,

$$\mathrm{E}\{\Delta a(t_2)\} = \lambda_1.a^2(t_2) - \lambda_2.a^2(t_2).\ \Delta^2\alpha_{máx} \tag{47}$$

for the uniform case and,

$$\mathrm{E}\{\Delta a(t_2)\} = \lambda_4.a^2(t_2) - \lambda_5.a^2(t_2).\sigma_\alpha^2 \tag{48}$$

for the gaussian case.

These results show once more the nonlinear relationship between cause and effect looked for. The terms $\lambda_1.a^2(t_2)$ and $\lambda_4.a^2(t_2)$ are constants and do not change the general form of the curves.

We can compare the both results of the deviations (uniform and gaussian) through the equation,

$$\Delta\alpha_{máx} = \sqrt{3}.\sigma_\alpha \tag{49}$$

If we replace this equation inside the (29), we obtain for the first order the results are the same, for the same σ_α and for others order, the gaussian semi-major axis deviations are $\frac{(2.n+1)!}{6^n.n!}$ more than the uniform deviations, for the same σ_α.

THE JPS1 CURVES OF NONIMPULSIVE AND COPLANAR ORBITAL MANEUVERS. NUMERICAL RESULTS

The numerical results confirm the algebraic results obtained. We simulated (Monte-Carlo) 1000 ensembles of the transfer trajectories for the both kind of deviations (uniform -U, gaussian - G), for the both maneuvers ("theoretical" - T, "practical" - P), for the random bias (S) and white noise (O) deviations. Figures 1 and 2 show E{a(t_2)} for cases TUS, TUO, TGS , TGO and PUS, PUO, PGS, PGO, respectively. In these figures DES2 = $\sqrt{3}.\sigma_{\Delta\alpha}$ where $\sigma_{\Delta\alpha}$ is the pitch angle standard deviation for zero mean.

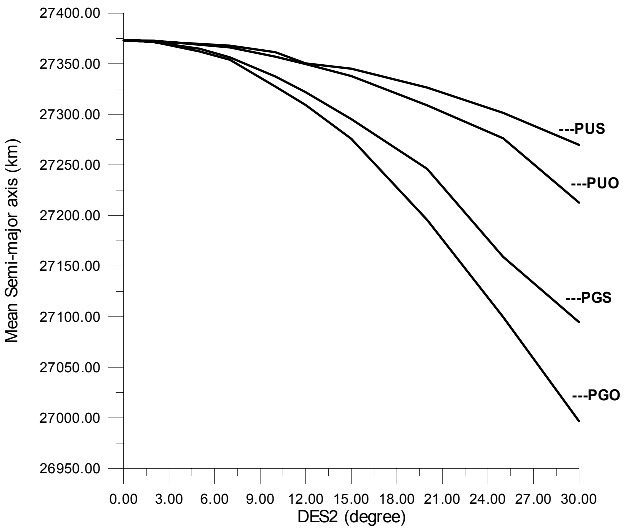

Fig. 1 – Mean Semi-major axis vs. DES2

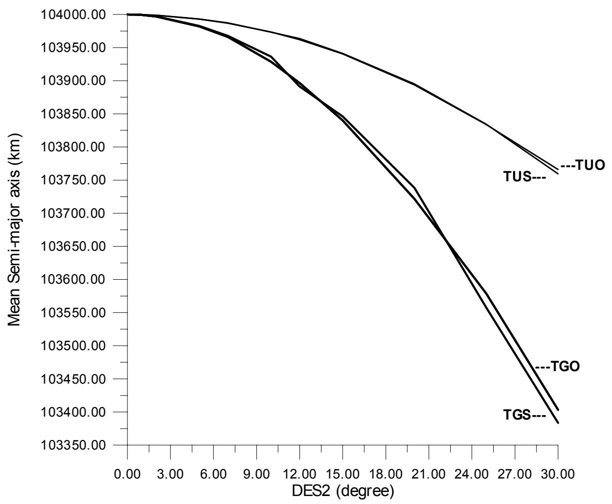

Fig. 2 – Mean Semi-major axis vs. DES2

CONCLUSIONS

In the algebraic developments, we obtained expressions for E{Δa(t_2)} as series of even powers of $\sigma_{\Delta\alpha}$ dominated by the $(\sigma_{\Delta\alpha})^2$ term, to explain the near parabolic relations and others found in the numerical phase, independent of the: 1) transfer orbit (theoretical or practical); 2) ensemble distribution (uniform or gaussian); 3) time correlation/dependence (random-bias or white-noise); 4) the gaussian deviations are more than the uniform deviations with mean linear coefficient between them equal 2.6 (numerical result) in all cases. The algebraic results anticipated the value 3.These results suggest and partially characterizes the progressive deformation of the trajectory distribution along the propulsive arc (JPS curves), turning 3-sigma ellipsoids into banana shaped volumes curved to the center of attraction (we call them "bananoids") due to the loss of optimality of the actual (with errors) trajectories with respect to the nominal (no errors) trajectory. Those results also characterize how close/far are Monte-Carlo analysis and covariance analysis for those examples.

REFERENCES

Adams, N. J., Melton, R.G. "Orbit Transfer Error Analysis for Multiple, Finite Perigee Burn, Ascent Trajectories". The Journal of Astronautical Sciences, 34(4), p. 355-373, 1986.

Alfriend, K. "Orbit Uncertainty Due to Drag Uncertainty". Paper presented at the 14th. International Symposium on Spacecraft Systems and Data Control – ISSSDC XIV at Iguassu Falls, Parana, Brazil, Febrary 8-12, 1999.

Biggs, M.C.B. "The Optimisation of Spacecraft Orbital Manoeuvres. Part I: Linearly Varying Thrust Angles". The Hattfield Polytechnic Numerical Optimisation Centre, Connecticut, USA, 1978.

Biggs, M.C.B. "The Optimisation of Spacecraft Orbital Manoeuvres. Part II: Using Pontryagin's Maximum Principle". The Hattfield Polytechnic Numerical Optimisation Centre, Connecticut, USA, 1979.

Carlton-Wippern, K. C. "Satellite Position Dilution of Precision (SPDOP)". AAS Paper 97-609, presented at the 1997 AAS/AIAA Astrodynamics Specialist Conference, Sun Vallley, ID, August 4-7, 1997.

Howell, K. C., Gordon, S. C. "Orbit Determination Error Analysis and a Station-Keeping Strategy for Sun-5-Earth L1 Libration Point Orbits". Journal of Astronautical Sciences, 42(2), p.207-228, 1994.

Jesus, A.D.C. "Análise Estatística de Manobras Orbitais com Propulsão Finita Sujeita a Erros no Vetor Empuxo". Doctoral Thesis. INPE, São José dos Campos, São Paulo, Brazil, 1999.

Jesus, A.D.C; Souza, M. L. O. and Prado, A. F. B. Monte-Carlo Analysis of Nonimpulsive Orbital Transfer Under Thrust Errors, 1. *Advances in the Astronautical Sciences*, USA, Vol. 103, No. 3, p.1923-1936, 2000. AAS/AIAA Astrodynamics Specialist Conference, Girdwood, Alaska, USA, August 16-19 (Paper AAS 99-424)

Jesus, A.D.C; Souza, M. L. O. and Prado, A. F. B. Monte-Carlo Analysis of Nonimpulsive Orbital Transfer Under Thrust Errors, 2. Nonlinear Dynamics, Chaos, Control and Their Applications to Engineering Sciences. Vol. 4: Recent Developments in Nonlinear Phenomena. Rio Claro, 2000, Vol. 4, p. 379-389.

Jesus, A.D.C.; Souza, M. L. O. and Prado, A. F. B. Statistical Analysis of Nonimpulsive Orbital Transfers under Thrust Errors, 1. Nonlinear Dynamics and Systems Theory. Vol. 2, n. 2, 2002, p. 157-172.

Junkins, J. L; Akella, M. R; Alfriend, K. T. Non-Gaussian Error Propagation in Orbital Mechanics. *The Journal of the Astronautical Sciences*, 44 (4), October-December, 541-563.

Junkins, J.L. "Adventures on the Interface of Dynamics and Control". *Journal of Guidance, Dynamics, and Control*, 20(6), p.1058-1071, 1997.

Kuga, H.K., Gill, E., Montenbruck, O. "Orbit Determination and Apogee Boost Maneuver Estimation Using UD Filtering". DLR-GSOC. Wesling, Germany, 1991 (Internal Report DLR-GSOC IB 91-2).

Porcelli, G., Vogel, E. "Two-Impulse Orbit Transfer Error Analysis Via Covariance Matrix". Journal of Spacecraft and Rockets 17(3), p. 248-255, 1980.

Prado, A.F.B.A. "Análise, Seleção e Implementação de Procedimentos que Visem Manobras Ótimas de Satélites Artificiais". Dissertação de Mestrado. INPE, São José dos Campos, São Paulo, Brazil, 1989 (INPE-5003-TDL/397).

Rao, K. R. Orbit Error Estimation in the Presence of Force Model Errors. AAS Paper 93-254, Advances in the Astronautical Sciences, Vol. 84, No. 1, pp. 61-74, 1993.

Rocco, E. M. *Transferências Orbitais Bi-impulsivas com Limite de Tempo e Erros no Vetor Empuxo*. Doctoral Thesis. 1999, S.J. Campos, SP, INPE.

Rodrigues, D.L.F. "Análise Dinâmica da Transferência Orbital". Master Dissertation. INPE, São José dos Campos, São Paulo, Brazil, 1991 (INPE-5352-TDI/461).

Santos-Paulo, M.M.N. "Estudo de Manobras Tridimensionais Impulsivas pelo Método de Altman e Pistiner, com Erros nos Propulsores". Master Dissertation. INPE, São José dos Campos, São Paulo, Brazil, 1998.
Schwende, M. A. and Strobl, H. Bi-propellant Propulsion Systems for Spacecraft Injection and Control, *Attitude and Orbit Control System*. Proceedings, ESA, Paris, 405-412, 1977.
Schuktz, W. Transferências Bi-impulsivas entre Órbitas Elípticas Não Coplanares com Consumo Mínimo de Combustível. Master Dissertation. INPE, 1997, São José dos Campos, SP, Brasil.
Tandon, G. K. Modeling Torques Due to Orbit Maneuvers, *Astrophysics and Space Science Library*, 733, 580-583, 1988.